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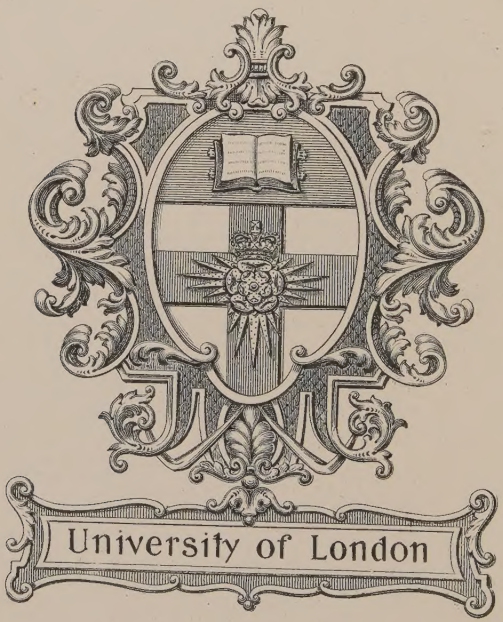

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. IX.

FROM NOVEMBER, 1877, TO NOVEMBER, 1878.

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LONDON MATHEMATICAL SOCIETY.

VOL. IX.

FOURTEENTH SESSION, 1877–78.

November 8th, 1877.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street,

Lord RAYLEIGH, F.R.S., President, in the Chair.

The Treasurer (S. Roberts, Esq., M.A.) read his Report, which, on the motion of Dr. Hirst, F.R.S., seconded by Major J. R. Campbell, was accepted. On the motion of Prof. Cayley, F.R.S., Mr. Perigal consented to act as Auditor.

The Secretaries' Report was read and accepted.

From the Report of the Secretaries, it appeared that the number of Members during the past Session was 134, and the number of Foreign Members 7, and that the present number of Life Members is 40.

The Obituary of the Society, during the Session, contained the name of Mr. Olinthus Gregory Downes.

The communications made to the Society during the past Session had been as follows :—

"On the Present State and Prospects of some Branches of Pure Mathematics:" Prof. H. J. S. Smith, F.R.S.

"On Curves having Four-point Contact with a Triply Infinite Pencil of Curves:" Dr. Spottiswoode, F.R.S.

"On some Classes of Multiple Definite Integrals:" Mr. E. B. Elliott, B.A.

"On certain Identical Differential Relations:" Mr. J. W. L. Glaisher, F.R.S.

"On the Conditions of Perpendicularity in a Parallelepipedal System:" Prof. H. J. S. Smith, F.R.S.

"On the Condition for the existence of a Surface cutting at Right Angles a given Set of Lines:" Prof. Cayley, F.R.S.

"The Orthogonal Transformation, and Additions to former Paper on Transformation of Elliptic Functions:" Prof. Clifford, F.R.S.

"On the Simplest Continuous Manifoldness of Two Dimensions and of Finite Extent:" Mr. Frankland.

"On the Theory of Electric Images, and its application to the case of Two Charged Spherical Conductors placed opposite one another:" Mr. W. D. Niven, M.A.

"On Viscous Fluids, and Quaternion Forms of some general Propositions in Fluid Motion:" Mr. J. G. Butcher, B.A.

"An easy method of finding the Invariant Equation expressing any Poristic Relation between Two Conics:" Prof. Wolstenholme, M.A.

"Determinant Conditions for Curves or Surfaces, of the same Order, having all their Intersections common:" Mr. J. Hammond, B.A.

"Numerical Values of the First Twelve Powers of π, of their Reciprocals, and of certain other Related Quantities:" Mr. J. W. L. Glaisher, F.R.S.

"On some General Classes of Multiple Definite Integrals:" Mr. E. B. Elliott, B.A.

"On the Partial Differential Equation $S+pP+Qq+Z=0$:" Prof. H. W. Lloyd Tanner, M.A.

"On Methods for finding Equations to Axes of a Conic, and the Axes themselves, in Trilinear Coordinates:" Mr. J. J. Walker, M.A.

"On some Elliptic Function Properties:" Prof. H. J. S. Smith, F.R.S.

"On the Area of the Quadrangle formed by the Four Points of Intersection of Two Conics:" Mr. C. Leudesdorf, B.A.

"On a Certain Series:" Mr. J. W. L. Glaisher, F.R.S.

"On the Equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$:" Prof. Cayley, F.R.S.

"Classification of Loci, and a Theorem in Residuation:" Prof. Clifford, F.R.S.

"On a New View of Pascal's Hexagram:" Mr. T. Cotterill, M.A.

"On a Class of Integers expressible as the Sum of Two Integral Squares:" Mr. T. Muir, M.A.

"Some Properties of the Double Theta Functions:" Prof. Cayley, F.R.S.

"Property of an Envelope;" and "On Hesse's Ternary Operator and Applications:" Mr. J. J. Walker, M.A.

"On the Circular Relation of Möbius;" and "On the Linear Transformation of the Integral $\int \frac{du}{\sqrt{U}}$:" Prof. Cayley, F.R.S.

"On some Cases of Parallel Motion:" Mr. H. Hart, M.A.

"A Method of Solving Partial Differential Equations which have a general First Integral, applied to Equations of the Third Order with Two Independent Variables:" Prof. H. W. Lloyd Tanner.

"Note on the Correlation of Two Planes:" Dr. Hirst, F.R.S.

"The Irreducible Case:" Herr Weichold.

"Generalization of Cases of Five-bar Motion considered at the April Meeting;" "The 'Kinematical Paradox';" and "A Method of Solving by Linkwork $f(x) = 0$, an Algebraical Equation of the n^{th} degree:" Mr. H. Hart.

"On the Free Motion of a Solid through an Infinite Mass of Perfect Liquid:" Prof. Lamb.

"On Mean Values:" Prof. M. W. Crofton, F.R.S.

"On the Canonical Form and Dissection of a Riemann's Surface:" Prof. Clifford, F.R.S.

"On Eisenstein's Theorem:" Prof. H. J. S. Smith, F.R.S.

"Proof that every Algebraic Equation has a Root:" Mr. J. C. Malet, M.A.

The following Mathematical Journals had been subscribed for:—

"Nouvelles Annales de Mathématiques." Paris.

"Giornale di Matematica." Napoli.

"Mathematische Annalen." Leipzig.

"Philosophical Magazine and Journal of Science." London.

"Quarterly Journal of Mathematics." Cambridge.

"Messenger of Mathematics." Cambridge.

The following Journals had been presented:—

"Crelle's Journal." Berlin.

"Monatsbericht." Berlin.

"Bulletin de la Société Mathématique de France." Paris.

"Bulletin des Sciences Mathématiques et Physiques." Paris.

"Annali di Matematica Pura ed Applicata." Milano.

"Proceedings of the Royal Society." London.

"Educational Times." London.

"Mathematical Reprint from the Educational Times." London.

"Journal of Education." London.

"Mémoires de la Société des Sciences physiques et naturelles de Bordeaux." Bordeaux.

"Jahrbuch über die Fortschritte der Mathematik." Berlin.

"Proceedings of the Physical Society of London." London.

"Atti della R. Accademia dei Lincei." Roma.

"Nieuw Archief voor Wiskunde." Amsterdam, 1876.

"Smithsonian Report." Washington, 1876.

"Memorie della R. Accademia di Scienze, Lettere ed Arti in Modena." Modena.

"Transactions of the Cambridge Philosophical Society." Cambridge.

"Journal of Institute of Actuaries." London.

The following presents had been made to the Society:—

Carte de Visite Likeness from Mr. A. B. Kempe, for the Album.

"Interpolation and Adjustment of Series," by E. L. de Forest. Newhaven, 1876.

"Lessons introductory to the Higher Algebra," by G. Salmon, D.D. Third Edition. Dublin, 1876.

"Weitere Untersuchungen über das Ikosaeder, i. ii. iii.," von F. Klein.

"Sur les équations qui se rencontrent dans la Théorie de la Transformation des Fonctions elliptiques," par le Père Joubert, S.J. Paris, 1876.

"Ueber das arithmetisch-geometrische Mittel aus vier Elementen," von C. W. Borchardt.

"Archiv for Mathematik og Naturvidenskab udgivet af Sophus Lie," W. Müller og G. O. Sars. 1876.

"Algemeine Theorie partielle Differential-Gleichungen 1·0," von S. Lie.

"Bidrag til Legemernes Molekylar-Theorie," af C. M. Guldberg.

"Om Attractionen mellem to Cirkel-flader," af O. Pihl.

"Discussion aller Integrations-Methoden der partiellen Differential-Gleichungen 1·0," von S. Lie.

"Sur la résolution des Equations du 2^{me}, 3^{me}, et 4^{me} degré par la fonction $\frac{n}{r}(X)$," par Dr. A. S. Guldberg.

(i.) "Zur Theorie des Integralitäts-Faktors." (ii.) "Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie," von S. Lie.

"Nautical Almanac" for 1880.

"Haversines, Natural and Logarithmic, used in computing Lunar Distances for the Nautical Almanac," edited by Major-General Hannyngton. 1876.

"Vorlesungen über Geometrie von A. Clebsch," bearbeitet und herausgegeben von Dr. F. Lindemann. Leipzig, 1876.

"Elementary Treatise on the Differential Calculus," by B. Williamson. M.A. 1877.

"Investigation of Corrections to Hansen's Tables of the Moon, with Tables for their Application," by S. Newcomb.

"Report on the Difference of Longitude between Washington and Ogden, Utah," by J. R. Eastman. Washington, 1876.

"Address delivered by J. J. Sylvester, F.R.S., at Johns Hopkins University, on Commemoration Day (Feb. 22, 1877)." Baltimore, 1877.

"Ueber den Verlauf der Abelschen Integrale bei den Curven vierten Grades," von F Klein.

"Elementary Treatise on the Integral Calculus," by B. Williamson, M.A. 1877.

"La Théorie Hugodécimale, ou la Base scientifique et définitive de l'Arithmologistique universelle," par le Cte. Léopold Hugo. Paris, 1877.

"Theory of Sound," Vol. i., by Lord Rayleigh, F.R.S.

"A Continuation of De Damoiseau's Tables of the Satellites of Jupiter to the year 1900," by D. P. Todd, B.A. Washington, 1876.

"Sull' equazione dell' Icosaedro nella risoluzione delle equazioni del quinto grado," nota del Prof. Klein.

"Astronomical and Meteorological Observations made during the year 1874 at the United States Naval Observatory," by Rear-Admiral C. H. Davis, U.S.N. Washington, 1877.

Catalogue of the Library of the University of London, including the Libraries of George Grote and A. de Morgan. London, 1876.

"Sur le paraboloide des huit droites," and "Sur les surfaces dont les rayons de courbure principaux sont fonctions l'un de l'autre," par M. A. Mannheim.

"ϕvelser i Grafisk Statik," af H. G. Zeuthen.

"Résumé du Bulletin de l'Académie Royale Danoise des Sciences et des Lettres pour l'année 1877."

"Lectures on the Elements of Applied Mechanics, comprising—1. Stability of Structures; 2. Strength of Materials;" by M. W. Crofton, F.R.S. 1877.

"Ueber correlative oder reciproke Bündel," von Rudolf Sturm.

"Ueber geometrische Erweiterungen des Bezoutschen Fundamentalsatzes," von Dr. H. Schubert, in Hamburg.

"Ueber lineare Differential-Gleichungen," von F. Klein.

"Sur la production du mouvement rectiligne exact, au moyen de tiges articulées," par A. B. Kempe (translated by M. V. Liguine).

"How to Draw a Straight Line," by A. B. Kempe, B.A.

"Sopra i sistemi tripli di superficie isoterme e ortogonali," Memoria di E. Betti.

"Sopra il moto di un sistema di un numero qualunque di punti," E. Betti.

"Boletin de la institucion libre de Enseñanza." Madrid, 1877.

The Meeting then proceeded to the election of the new Council. The Scrutators (Prof. Greenhill and Mr. R. Rawson), having examined

the balloting lists, declared the following gentlemen (nominated by the Council) duly elected :—

President—The Right Hon. Lord Rayleigh, F.R.S. Vice-Presidents —Prof. J. Clerk-Maxwell, F.R.S.; Mr. C. W. Merrifield, F.R.S.; Prof. H. J. S. Smith, F.R.S. Treasurer—Mr. S. Roberts, M.A. Hon. Secretaries—Messrs. Morgan Jenkins, M.A., and R. Tucker, M.A. Other Members—Prof. Cayley, F.R.S.; Mr. T. Cotterill, M.A.; Mr. J. W. L. Glaisher, F.R.S.; Mr. Harry Hart, M.A.; Dr. Henrici, F.R.S.; Dr. Hirst, F.R.S.; Mr. A. B. Kempe, B.A.; Dr. Spottiswoode, F.R.S.; Mr. J. J. Walker, M.A.

Rev. William Ellis, B.A., Caius College, Cambridge, was proposed for election.

The following communications were made to the Society :—

"On the Function $\phi(x) = \frac{ax+b}{cx+d}$," and "On the Theta Functions:" Prof. Cayley.

"The Calculus of Equivalent Statements:" Hugh McColl, B.A. (communicated by Prof. Crofton, read by Mr. Tucker).

"Progressive Waves:" Lord Rayleigh.

Prof. Clifford communicated notes "On the Triple Generation of Three-bar Curves," "On the Mass-Centre of an Octahedron," and "On Vortex Motion."

The following presents were made to the Society :—

"The Optical Department of the Atmosphere in relation to the Phenomena of Putrefaction and Infection," by J. Tyndall, F.R.S. (Phil. Trans., Part i. 1876, Vol. 166; read Jan. 13, 1876.)

"On the Atmosphere as a Vehicle of Sound," by the same. (Read Feb. 12th, 1874.)

"On the Variations of the Daily Mean Horizontal Force of the Earth's Magnetism produced by the Sun's Rotation and the Moon's Synodical and Tropical Revolutions," by J. A. Broun, F.R.S. (Vol. 166, Part 2; read Jan. 27, 1876.)

"On a Class of Identical Relations in the Theory of Elliptic Functions," by J. W. L. Glaisher, M.A. (Vol. 165, Part 2; read Jan. 14, 1875.)

"On the Mathematical Theory of Stream-Lines, especially those with four Foci and upwards," by W. J. M. Rankine, C.E., &c. (Read Feb. 10, 1870); with Supplement.

"Éloge de MM. Alexandre Brongniart et Adolphe Brongniart," par M. Dumas (Institut de France, Paris, 1877).

"Détermination nouvelle de la Vitesse de la Lumière," par M. A. Cornu (Journal de l'École Polytechnique, xliv^{e} cahier); "Sur la détermination de la Vitesse de la Lumière," par M. A. Cornu (Comptes Rendus, 2 Oct. 1871); "Détermination nouvelle de la Vitesse de la

Lumière" (Comptes Rendus, 10 Fevr. 1873); "Détermination de la Vitesse de la Lumière et de la Parallaxe du Soleil," par M. Cornu (Comptes Rendus, Dec. 14, 1874).

"Détermination, par le principe de Correspondance, de la classe de la développée et de la caustique par réflexion d'une courbe géométrique d'ordre m et de la classe n," par M. Chasles (Comptes Rendus, t. lxxii. 1871).

"Propriétés des systèmes de coniques, relatives, toutes à certaines séries de normales en rapport avec d'autres lignes ou divers points" (Do. do.)

"Propriétés des systèmes de coniques, dans lesquels se trouvent des conditions de perpendicularité entre diverses séries de droites" (Do. do.)

"Théorèmes divers concernant les systèmes de coniques représentés par deux caractéristiques" (Do. do.)

"Démonstration géométrique d'une proposition due à M. Bertrand," par M. A. Mannheim (Journal de Mathématiques pures et appliquées, 2e Série, Tome xvii. 1872).

"Sur la surface gauche, lieu des normales principales de deux courbes" (with the last).

"Mémoire sur les pinceaux de droites et les normaliés contenant une nouvelle exposition de la théorie de la courbure des surfaces" (same work), par M. A. Mannheim.

"Exposition sommaire d'une théorie géométrique de la courbure des surfaces" (Comptes Rendus, 26 Fev. 1872), par M. Mannheim.

"Recherches géométriques sur le contact du 3e ordre de deux surfaces" (Do. do., 25 Mars, 1872).

"Démonstration géométrique de quelques théorèmes, au moyen de la considération d'une rotation infiniment petite" (Do. do., 2 Mars, 1874).

"Deux théorèmes nouveaux sur la surface de l'onde" (Do. do., 23 Mars, 1874).

"Construction directe du centre de courbure en un point de la section faite dans une surface par un plan quelconque" (Do. do., 6 Avr. 1874).

"Détermination des relations analytiques qui existent entre les éléments de courbure des deux nappes de la développée d'une surface" (Do. do., 7 Dec. 1874).

"Solutions géométriques de quelques (et de nouveaux) problèmes, relatifs à la théorie des surfaces et qui dépendent des infiniment petits du troisième ordre" (Do. do., 1 Mars, 22 Mars, 1875).

"Sur la résolution des équations les unes par les autres," par M. C. Jordan (Do. do., 13 Mars, 1871).

"Théorèmes sur les groupes primitifs," par M. C. Jordan (Do. do., 26 Juin, 1871).

"Mémoire sur la résolution des équations algébriques les unes par les autres," par M. C. Jordan (extrait du Journal des Mathématiques pures et appliquées," 2e Série, Tome xvi. 1871).

"La Revue Scientifique de la France et de l'étranger revue des cours scientifiques" (2e Série). No. 10, 2nd Sept. 1871, contains "Rapports des sciences physiques avec les sciences mathématiques—Théorie atomique, Prof. J. Clerk-Maxwell (British Association, Liverpool)," and "La polarisation rotatoire—expériences de Sir Ch. Wheatstone, Mr. Spottiswoode (Royal Institution, Friday Lecture)."

Do. do., No. 47, 24 Mai, 1873, "Histoire des Laboratoires de l'Institution Royale, Mr. Spottiswoode (Royal Institution, Friday Lecture)."

Do. do., No. 20, 2e Série, 14 Nov. 1874, "Les combinaisons de couleurs produites au moyen de la lumière polarisée, Mr. Spottiswoode (Royal Institution, Friday Lecture)."

(All the above were presented by Dr. Spottiswoode.)

"Annali di Matematica," Serie iia, Tomo viii°, Fasc. 3° (Ottobre, 1877); Milano.

"Educational Times," Dec. 1877.

"Smithsonian Report," 1876; Washington, 1877.

"Monatsbericht," Aug. 1877; Berlin.

"Nautical Almanac" for 1881, from Lords Commissioners of the Admiralty.

"Elementary Theorems relating to the Geometry of a Space of Three Dimensions, and of Uniform Positive Curvature in the Fourth Dimension," by Simon Newcomb, Washington, U.S. (Journal für Mathematik, Band lxxxiii., Heft 4, pp. 293–9): from the Author.

"Quatre modèles représentant des surfaces développables avec des renseignements sur la construction des modèles et sur les singularités qu'ils représentent, par V. Malthe Bruun et C. Crone, avec quelques remarques sur les surfaces développables et sur l'utilité des modèles, par M. le Dr. H. G. Zeuthen," Copenhague, 1877: from the Authors.

"On Knots," by Prof. Tait (Trans. R. S. Edin., Vol. xxviii., Part i., pp. 145—190, and Plates): from the Author.

"A List of Writings relating to the Method of Least Squares, with Historical and Critical Notes," by Mansfield Merriman, Ph.D. (from the Transactions of the Connecticut Academy, Vol. iv., 1877): from the Author.

The Calculus of Equivalent Statements and Integration Limits.

By Hugh McColl, B.A.

[*Read Nov.* 8, 1877.]

The above title seems to be the most suitable for an analytical method which I discovered a few months ago, and to which a short introduction was published in the "Educational Times" for last July, under the name of "Symbolical Language." The chief use of the method, as far as I have yet carried it, is to determine the new limits of integration when we change the order of integration or the variables in a multiple integral, and also to determine the limits of integration in questions relating to probability. This object it will accomplish with perfect certainty, and by a process almost as simple and mechanical as the ordinary operations of elementary algebra. The fundamental principles of the method are as follows:

Definition 1.—Let any symbols, say A, B, C, &c., denote *statements* (or propositions) registered for convenience of reference in a table. Then the equation $A=1$ asserts that the statement A is *true;* the equation $A=0$ asserts that the statement A is *false;* and the equation $A=B$ asserts that A and B are equivalent statements.

Def. 2.—The symbol $A\times B\times C$ or ABC denotes a *compound* statement, of which the statements A, B, C may be called the *factors.* The equation $ABC=1$ asserts that *all the three statements are true;* the equation $ABC=0$ asserts that all the three statements are *not* true, *i. e.*, that at least *one* of the three is false. Similarly a compound statement of any number of factors may be defined.

Def. 3.—The symbol $A+B+C$ denotes an *indeterminate* statement, of which the statements A, B, C may be called the *terms.* The equation $A+B+C=0$ asserts that all the three statements are *false;* the equation $A+B+C=1$ asserts that all the three are *not* false, *i. e.*, that at least *one* of the three is true. Similarly an indeterminate statement of any number of terms may be defined.

Def. 4.—The symbol A' is the *denial* of the statement A. The two statements A and A' are so related that they satisfy the two equations $A+A'=1$ and $AA'=0$; that is to say, one of the two statements (either A or A') must be true and the other false. The same symbol (*i. e.*, a *dash*) will convert any complex statement into its denial. For example, $(AB)'$ is the denial of the compound statement AB.

Note.—The statements A and A' are what logicians call "contradictories"; and the two equations $A+A'=1$ and $AA'=0$ combined express the principle known in logic as the "Law of Excluded Middle."

Def. 5.—When only *one* of the terms of an indeterminate statement $A+B+C+\ldots$ can be true, or when no two terms can be true at the same time, the terms are said to be *mutually inconsistent* or *mutually exclusive.*

Rule 1.—The rule of ordinary algebraical multiplication applies to the multiplication of indeterminate statements, thus :

$$A(B+C) = AB+AC; \quad (A+B)(C+D) = AC+AD+BC+BD;$$

and so on for any number of factors, and whatever be the number of terms in the respective factors.

Note.—It is evident that if the terms of every indeterminate factor be mutually inconsistent, the terms of the product will also be mutually inconsistent.

Rule 2.—Let A be any statement whatever, and let B be any statement which is implied in A (and which must therefore be true when A is true, and false when A is false); or else let B be any statement which is admitted to be true independently of A; then (in either case) we have the equation $A = AB$. As particular cases of this we have $A = AA = AAA =$ &c., as repetition neither strengthens nor weakens the logical value of a statement. Also,

$$A = A(B+B') = A(B+B')(C+C') = \text{\&c.},$$

for $B+B' = 1 = C+C' =$ &c. (See Def. 4.)

Rule 3.
$$\begin{aligned}(AB)' &= AB'+A'B+A'B' \\ &= AB'+A'(B+B') = AB'+A' \\ &= A'B+B'(A+A') = A'B+B',\end{aligned}$$

for $A+A'=1$ and $B+B'=1$. Similarly we may obtain various equivalents (with mutually inconsistent terms) for $(ABC)'$, $(ABCD)'$, &c.

Rule 4. $(A+B)' = A'B'$; $(A+B+C)' = A'B'C'$;

and so on.

Rule 5.
$$\begin{aligned}A+B &= \{(A+B)'\}' = (A'B')' \\ &= AB'+A'B+AB \\ &= AB'+(A'+A)B = AB'+B \\ &= A'B+A(B'+B) = A'B+A.\end{aligned}$$

Similarly we get equivalents (with mutually inconsistent terms) for $A+B+C$, $A+B+C+D$, &c.

The foregoing principles constitute the elementary *basis* of the method. We now come to the more important part of the subject, namely, the application of the method to multiple integrals and probability.

Def. 6.—The symbol p prefixed to any algebraical (or arithmetical) expression converts the expression into a statement, namely, that the expression is *real and positive;* the symbol p' in like manner asserts that the expression affected by it is *real and negative.* For example, if we know that x and y are both real, we have the equations:

$$p(xy) = px\,py + p'x\,p'y,$$
$$p'(xy) = p'x\,py + px\,p'y.$$

Again, suppose we know that x, y, a are all three real and positive, it is easy to see the identities,

$$p(x^2+y^2-a^2) = p\{y-\surd(a^2-x^2)\}\,p'(x-a) + p(x-a),$$
$$p'(x^2+y^2-a^2) = p'\{y-\surd(a^2-x^2)\}\,p'(x-a).$$

Note.—We might also use a symbol q to denote the statement that the expression affected by it was imaginary, but I do not think that the need for the symbol would often arise.

Def. 7.—The symbols px, $p'x$, py, $p'y$, &c. occur so frequently that it is convenient to replace them respectively by x_0, $x_{0'}$, y_0, $y_{0'}$, &c. Another reason for the employment of these last symbols will appear later.

Def. 8.—The symbols x_1, x_2, x_3, &c. denote the 1st, 2nd, 3rd, &c. limits of x registered in any convenient order in a table of reference. The limits of the other variables of the expression (or expressions) under consideration are denoted similarly.

Note.—Among the limits of the variables thus registered, the limit *zero* is not included (see Defs. 7 and 9); but we may denote it either by the usual symbol 0, or (for the sake of uniformity in the notation) by any of the symbols x_0, y_0, z_0, &c.

Def. 9.—The symbols x_1, x_2, x_3, &c. also denote *statements*, namely, the statements that the *limits* x_1, x_2, x_3, &c. are *inferior* limits of x. Similarly y_1, y_2, &c., z_1, z_2, &c. are to be interpreted.

Note.—The symbol x_m has thus two meanings: it denotes the m^{th} *limit* of x, and it also denotes the *statement* that this limit is an *inferior limit;* in other words, x_m is an abbreviation for $p(x-x_m)$. The context will always prevent any confusion of ideas resulting from this double signification of the same symbol.

Def. 10.—The symbols $x_{1'}$, $x_{2'}$, $x_{3'}$, &c. denote the *statements* that the limits x_1, x_2, x_3, &c. are *superior* limits of x. Similarly $y_{1'}$, $y_{2'}$, &c., $z_{1'}$, $z_{2'}$, &c. are to be interpreted.

Note.—The symbol $x_{m'}$ is thus an abbreviation for the statement $p'(x-x_m)$.

Def. 11.—The symbol $x_{m'n'rs}$ is an abbreviation for the symbol $x_{m'}x_{n'}x_rx_s$, and denotes the compound statement

$$p'(x-x_m)\,p'(x-x_n)\,p(x-x_r)\,p(x-x_s).$$

Similarly a compound statement of any number of factors, and having reference to the limits of x or of any other variable, may be abbreviated.

Rule 6.—The compound statement

$$x_{m'n'r's'\ldots} = x_{m'}\alpha + x_{n'}\beta + x_{r'}\gamma + x_{s'}\delta + \ldots,$$

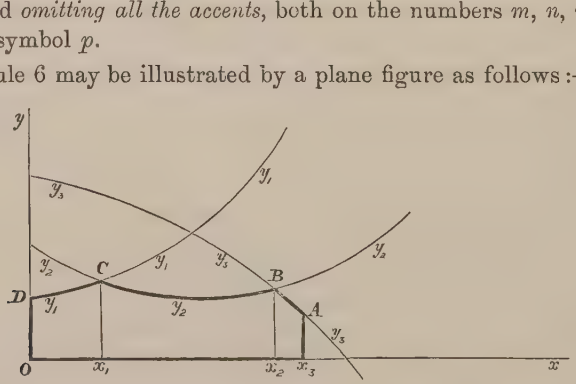
in which $x_{m'}$, $x_{n'}$, &c. are abbreviations for the statements that the m^{th}, n^{th}, &c. limits of x are all *superior* limits; while α, β, γ, &c. respectively denote the statements that amongst these x_m is the nearest superior limit of x, that x_n is the nearest superior limit, that x_r is the nearest superior limit, and so on. In other words, α is an abbreviation for the compound statement

$$p'(x_m-x_n)\,p'(x_m-x_r)\,p'(x_m-x_s)\ldots.$$

The value of β is obtained from this expression by simply interchanging m and n; the value of γ is obtained from the expression for β by interchanging n and r; and so on.

Rule 7.—This is obtained from the preceding Rule by simply copying all the words in it (except *superior*, for which we must write *inferior*), and *omitting all the accents*, both on the numbers m, n, r, s, ... and on the symbol p.

Note.—Rule 6 may be illustrated by a plane figure as follows:—



Let $y=\phi(x)$, $y=\psi(x)$, $y=\chi(x)$ be the three equations for the curves marked respectively $y_1y_1y_1$, $y_2y_2y_2$, $y_3y_3y_3$. Then all the points contained within the thick boundary DOx_3ABCD will be expressed by the statement $y_{1'.2'.3'.0}x_{3'.0}$;

and this statement is evidently equivalent to the statement

$$y_{1'.0}x_{1'.0}+y_{2'.0}x_{2'.1}+y_{3'.0}x_{3'.2},$$

x and y being the Cartesian coordinates of any point whatever within the boundary.

A similar geometrical illustration may be given of Rule 7.

RULE 8.—The statement $x_{m'n} = x_{m'n}\,\alpha$, in which $\alpha = p\,(x_m - x_n)$.

Note.—This is a particular case of Rule 2.

Rules 6 and 7 may also be brought under Rule 2 as follows:— Attaching the same meaning to α, β, γ, &c. as in Rule 6, it is evident that $\alpha+\beta+\gamma+\delta+\ldots = 1$. Hence, by Rule 2, we get

$$x_{m'n'r's'\ldots} = x_{m'n'r's'\ldots}\,(\alpha+\beta+\gamma+\delta+\ldots)\,;$$

and since $x_{m'}\,\alpha = x_{m'n'r's'\ldots}\,\alpha$, and so on for the β, γ, δ, &c. terms, we get Rule 6 by the suppression of implied factors. We may similarly show Rule 7 to be a particular case of Rule 2.

The last three Rules, 6, 7, 8 (combined with Rule 1), constitute the pivot on which the whole process turns, whether in its application to the transformation of multiple integrals or to probability. By repeated application of these three Rules to the several variables in succession, any compound statement of the form

$$x_{1'.\,2'.\,0'\ldots\,3\,.\,4\,.\,6\ldots}\; y_{5'.\,7'.\,2'\ldots\,0\,.\,3\,.\,4\ldots}\; z_{3'.\,4'.\,0'\ldots\,1\,.\,2\,.\,6\ldots}\;\cdots,$$

with any number of variables, and any number of factors for each variable, will finally be reduced either to a single *elementary** term of the form $x_{m'n}\,y_{r's}\,z_{t'u}\ldots$, in which x_m and x_n are the *nearest* limits (superior and inferior) of x, and so on for the variables y, z, &c.; or (as will generally happen in complicated cases) it will be reduced to an indeterminate statement consisting of several such terms. The work may generally be much abbreviated by dropping *zero terms* (*i. e.*, terms with inconsistent factors) as we go along, when mere inspection of the table of limits (without having recourse to Rule 8) will suffice to detect them. But if we overlook these zero terms, they will eventually disappear of themselves in the subsequent evolutions of the process. We may also shorten the process by cancelling factors which mere inspection of the table (instead of having recourse to Rule 6 or 7) will show to be implied in their co-factors *of the same variable.*

To give a practical illustration of the method, we will take a problem of some complexity. Suppose we are required to *reverse the order of integration* in the multiple integral

$$\int_{-a}^{2a} du \int_{-u}^{2u} dx \int_{-x}^{2x} dy \int_{-2x}^{\frac{y^2}{2x}} dz\;\phi\,(u, x, y, z).$$

* A term of this form may be called *elementary* when the application of Rule 8 will introduce no fresh factor except unity, or such factors as Rules 6 or 7 would afterwards reject as unnecessary.

For greater facility of reference throughout the process, the annexed table of limits may be conveniently made out on a card or moveable slip of paper. The values severally entered in the table during the course of the operations are found to arise spontaneously in reducing the various factors to convenient symbols. Each limit is registered in the table as soon as it is ascertained, so that the table *grows* as the process proceeds. Sometimes a limit which has already been registered as x_m may again inadvertently be registered as x_n; when this happens the oversight will be detected later by the appearance of an anomalous statement, such as

$$p\,(x_m - x_n) = p\,(0).$$

Table of Limits.

$u_1 = -a$	$x_1 = -y$	$y_1 = \frac{1}{2}z$	$z_1 = -8a$
$u_2 = 2a$	$x_2 = \frac{1}{2}y$	$y_2 = -z$	$z_2 = 2a$
$u_3 = -x$	$x_3 = -\frac{1}{2}z$	$y_3 = \sqrt{8az}$	$z_3 = 8a$
$u_4 = \frac{1}{2}x$	$x_4 = \frac{y^2}{2z}$	$y_4 = -\sqrt{8az}$	
	$x_5 = a$	$y = -2z$	
	$x_6 = -2a$	$y_6 = -4a$	
	$x_7 = 4a$	$y_7 = z$	
		$y_8 = 8a$	

From the integral we get a compound statement of 8 factors (2 for each variable); so that, if we denote the compound statement by A, and the 8 factors by A_1, A_2, A_3, &c., we have

$$\begin{aligned}
A_1 &= p\,(u + a) = u_1,\\
A_2 &= p'(u - 2a) = u_{2'},\\
A_3 &= p\,(x + u) = u_3,\\
A_4 &= p'(x - 2u) = p\,(u - \tfrac{1}{2}x) = u_4,\\
A_5 &= p\,(y + x) = x_1,\\
A_6 &= p'(y - 2x) = p\,(x - \tfrac{1}{2}y) = x_2,\\
A_7 &= p\,(z + 2x) = p\,(x + \tfrac{1}{2}z) = x_3,\\
A_8 &= p'\left(z - \frac{y^2}{2x}\right) = p'\left\{\frac{(x - x_4)\,z}{x}\right\}.
\end{aligned}$$

But evidently $p'\left(\dfrac{x - x_4}{x}\right) = x_{4'.0} + x_{0'.4}$; $pz = z_0$;

$$p\left(\frac{x - x_4}{x}\right) = x_{4.0} + x_{4'.0'};\quad p'z = z_{0'};$$

and therefore $$\begin{aligned} p'\left\{\frac{x - x_4}{x}z\right\} &= (x_{4'.0} + x_{0'.4})\,z_0 + (x_{4.0} + x_{4'.0'})\,z_{0'}\\ &= x_{4'.0}\,z_0 + x_0\,z_{0'} + x_{4'}\,z_{0'}, \end{aligned}$$

for by inspection of the table of limits we see that $x_{0'.4}\,z_0 = 0$, $x_{4.0}\,z_{0'} = x_0\,z_{0'}$, $x_{4'.0'}\,z_{0'} = x_4 z_{0'}$. Hence we have

$$A = u_{2'.1.3.4}\,x_{1.2.3}\,(x_{4'.0}\,z_0 + x_0\,z_{0'} + x_{4'}\,z_{0'}).$$

Now the simplest and generally the most advantageous order in the application of Rules 6, 7, 8 would be to apply these rules first to the u-factors; then (after multiplying) to the x-factors; then to the y-factors which would arise; and lastly to the z-factors. But a little saving of labour will be effected in this instance by slightly departing from this order, which is never an absolutely necessary order of application. We will first apply Rule 7 to the compound statements $u_{1.3.4}$ and $x_{1.2.3}$, and then Rule 8 to the u-factors, thus:

$$u_{1.3.4} = u_1\alpha + u_3\beta + u_4\gamma,$$

in which

$$\begin{aligned}
\alpha &= p\,(u_1 - u_3)\;\; p\,(u_1 - u_4) \\
&= p\,(-a+x)\; p\,(-a-\tfrac{1}{2}x) = x_{5.6'} = 0, \\
\beta &= p\,(u_3 - u_1)\;\; p\,(u_3 - u_4) \;\; = x_{5'.0'} = x_{0'}, \\
\gamma &= p\,(u_4 - u_1)\;\; p\,(u_4 - u_3) \;\; = x_{6.0} = x_0.
\end{aligned}$$

Expanding the compound statement $x_{1.2.3}$ in the same way into an indeterminate statement, and substituting, we get

$$A = u_{2'}\,(u_3 x_{0'} + u_4 x_0)\,(x_1 y_{0'.1'} + x_2 y_{0.2} + x_3 y_{2'.1})\,(x_{4'.0} z_0 + x_0 z_{0'} + x_{4'} z_{0'}).$$

Multiplying the three indeterminate factors, omitting the zero terms in the result, and cancelling those factors in each term which mere inspection of the table will show to be implied in their co-factors of the same variable, we get (see Appendix, Note α)

$$A = u_{2'.4}\,(x_{4'.1} y_{0'} + x_{4'.2} y_0)\, z_0 + u_{2'.4}\,(x_1 y_{1'} + x_2 y_2 + x_3 y_{2'.1})\, z_{0'}.$$

Applying now Rule 8 to the compound factor $u_{2'.4}$, we get

$$u_{2'.4} = u_{2'.4}\, x_{7'},$$

so that a fresh factor $x_{7'}$ is introduced among the factors in x (see Appendix, Note β).

We have now done with the limits of the variable u, so we apply our rules (when mere inspection of the table is not sufficient) to the x-statements, and we get (after cancelling implied factors in y, see Appendix, Note β)

$$\begin{aligned}
A = \;& u_{2'.4}\,(x_{4'.1} y_{5'.4} + x_{7'.1} y_{4'.6} + x_{4'.2} y_{3'.7} + x_{7'.2} y_{3'.3})\, z_0 \\
&+ u_{2'.4}\,(x_{7'.1} y_{1'.6} + x_{7'.2} y_{3'.2} + x_{7'.3} y_{2'.1} z_1)\, z_{0'}.
\end{aligned}$$

Having now done with the limits of u and x, we apply Rule 8 (Rules 6 and 7, not being required as implied factors in y, have already been cancelled) to the y statements, when we shall get finally

$$\begin{aligned}
A = \;& z_{2'.0}\,(y_{5'.4} x_{4.1} + y_{4'.6} x_{7.1})\, u_{2'.4} \\
&+ z_{3'.0}\,(y_{3'.7} x_{4'.2} + y_{3'.3} x_{7'.2})\, u_{2'.4} \\
&+ z_{0'.1}\,(y_{1'.6} x_{7'.1} + y_{3'.2} x_{7'.2} + y_{2'.1} x_{7'.3})\, u_{2'.4},
\end{aligned}$$

altogether 7 elementary terms. The first is $z_{2'.0}y_{5'.4}x_{4'.1}u_{2'.4}$, the corresponding term of the transformed integral being

$$\int_0^{z_2} dz \int_{y_4}^{y_5} dy \int_{x_1}^{x_4} dx \int_{u_4}^{u_2} du\ \phi(u, x, y, z);$$

and so on for the remaining 6 terms and the corresponding **6** terms of the transformed integral.

The mode of applying the process to find the limits of integration when we change the *variables* in a multiple integral is so obvious from the mode of applying it in finding the limits when we change the *order of integration*, that it is unnecessary to illustrate it by a separate example. We shall therefore end this article by applying the method to an easy question in probability.

In the quadratic equation $x\theta^2 - y\theta + z = 0$, if the coefficients x, y, z be each taken at random between a and 0, what is the chance that the roots of the equation will be real, all values of x, y, z between the given limits being equally probable?

Let A denote the statement whose truth is taken for granted, namely, the statement $x_{1'.0}y_{1'.0}z_{1'.0}$ (see the table); and let Q denote the statement which may be true or false, namely, $p'(y^2 - 4xz)$. Then the required chance is

$$\frac{AQ\iiint dx\,dy\,dz}{A\iiint dx\,dy\,dz},$$

the statements A and AQ fixing the limits of integration for the denominator and numerator respectively.

The denominator of the above fraction is evidently

$$\int_0^{x_1} dx \int_0^{y_1} dy \int_0^{z_1} dz = a^3,$$

the statement A being elementary, since the application of rule 8 will introduce no fresh factors. It remains to find the limits of integration for the numerator from the statement AQ.

TABLE OF LIMITS.

$z_1 = a$	$y_1 = a$	$x_1 = a$
$z_2 = \dfrac{y^2}{4x}$	$y_2 = \sqrt{4ax}$	$x_2 = \dfrac{a}{4}$

Now $Q = p'(4xz - y^2)$, and since x is positive, this

$$= p'\left(z - \frac{y^2}{4x}\right) = z_{2'}.$$

Hence $$AQ = x_{1'.0}y_{1'.0}z_{2'.1'.0}.$$

But $$z_{2'.1'} = z_{2'}y_{2'} + z_{1'}y_2, \quad \text{by Rule 6.}$$

Hence $$AQ = x_{1'.0}\,y_{1'.0}\,(z_{2'}\,y_{2'} + z_{1'}\,y_2)\,z_0$$
$$= x_{1'.0}\,(y_{2'.1'.0}\,z_{2'.0} + y_{1'.0.2}\,z_{1'.0}).$$

But $y_{2'.1'} = y_{2'} x_{2'} + y_{1'} x_2$ by Rule 6, and $y_{0.2} = y_2$ by Rule 7 or mere inspection of the table.

Hence $AQ = x_{2'.1'.0}\, y_{2'.0}\, z_{2'.0} + x_{1'.0.2}\, y_{1'.0}\, z_{2'.0} + x_{1'.0}\, y_{1'.2}\, z_{1'.0}$.

But by mere inspection of the table we get $x_{2'.1'} = x_{2'}$ and $x_{0.2} = x_2$; and in the third term we get (by Rule 8) $y_{1'.2} = y_{1'.2}\, x_{2'}$.

Hence, finally,

$$AQ = x_{2'.0}\, y_{2'.0}\, z_{2'.0} + x_{1'.2}\, y_{1'.0}\, z_{2'.0} + x_{2'.0}\, y_{1'.2}\, z_{1'.0}.$$

These three terms are elementary, since rule 8 will introduce no fresh factors. Hence

$$AQ \iiint dx\,dy\,dz = \int_0^{x_2} dx \int_0^{y_2} dy \int_0^{z_2} dz + \int_{x_2}^{x_1} dx \int_0^{y_1} dy \int_0^{z_2} dz + \int_0^{x_2} dx \int_{y_2}^{y_1} dy \int_0^{z_1} dz.$$

The integrations are easy, and the result is $(\frac{5}{36} + \frac{1}{6} \log_e 2)\, a^3$.

The required chance is therefore $\frac{5}{36} + \frac{1}{6} \log_e 2$.

With reference to the preceding solution, the referees of the Mathematical Society have kindly made the following valuable suggestion, which may also be extended to other problems:

"The process will be considerably abbreviated and simplified if from the outset the statements x_0, y_0, z_0, $x_{1'}$, $y_{1'}$, $z_{1'}$ are severally regarded as *unit*-factors, and therefore omitted when not wanted. Thus the whole working would be as follows:

$$AQ = z_{2'} = z_{2'.1'} = z_{2'}\, y_{2'} + z_{1'}\, y_2$$
$$= z_{2'}\, y_{2'.1'} + y_2 = z_{2'}\, (y_{2'}\, x_{2'} + y_{1'}\, x_2) + y_2.$$

And, by restoring or supplying the proper unit-factors, the final result is at once obtained."

In accordance with this suggestion I would propose the following convention:—

When we are analyzing any factor A of any compound statement $ABC\ldots$, the truth of its co-factors B, C, ... may for the time be taken for granted, so that as *unit-factors* they may be introduced or suppressed at pleasure. The equation $A = a$, according to this convention, will assert, *not* that a is *always* an equivalent for A, but *that a may replace A in the particular compound statement of which A is a factor.* We may then have such equations as $A = AB = aB = a = aC =$ &c. According to this convention, it is evident that $A = 1$ asserts either that A is *implied* in some co-factor (which co-factor may be true or false), or else that A is *true absolutely* and independently of any co-factor. Also $A = 0$ asserts either that A is *inconsistent* with some co-factor (which co-factor may be true or false), or else that A is *false absolutely* and independently of any co-factor.

These abbreviations, however, necessitate more caution in the working of the process: for, on the one hand, care is needed lest any factor, left temporarily understood for the sake of brevity, should inadvertently be left out of account altogether; and, on the other hand, lest factors which have been already taken into account should again be needlessly introduced when their services are no longer required. For these reasons I think the abbreviations cannot be employed with safety till some familiarity has been first acquired with the longer but easier method adopted in the text.

Appendix.

Note *a*.—This is obtained as follows:—Multiplying the last two indeterminate factors, namely,

$$x_1 y_{0'.1'} + x_2 y_{0.2} + x_3 y_{2'.1},$$

and

$$x_{4'.0} z_0 + x_0 z_{0'} + x_{4'} z_{0'},$$

we get for our product

$$\begin{aligned} & x_{4'.1.0}\ y_{0'.1'}\ z_0 + x_{4'.2.0}\ y_{0.2}\ z_0 + \underline{x_{4'.3.0}\ y_{2'.1}\ z_0} \\ + & x_{1.0}\ \ y_{0'.1'}\ z_{0'} + x_{2.0}\ \ y_{0.2}\ z_{0'} + x_{3.0}\ \ y_{2'.1}\ z_{0'} \\ + & x_{4'.1}\ \ y_{0'.1'}\ z_{0'} + x_{4'.2}\ \ y_{0.2}\ z_{0'} + x_{4'.3}\ \ y_{2'.1}\ z_{0'}. \end{aligned}$$

But the third term (the one underlined) is zero, since it contains the inconsistent compound factor $y_{2'.1} z_0$; for (by Rule 8)

$$y_{2'.1} = y_{2'.1}\, p(y_2 - y_1) = y_{2'.1}\, p(-z - \tfrac{1}{2}z) = y_{2'.1} z_{0'},$$

and $z_{0'}$ is inconsistent with z_0.

Omitting the underlined term therefore, and multiplying the terms left by $u_3 x_{0'} + u_4 x_0$, we get (omitting the terms which contain the inconsistent factor $x_{0'.0}$)

$$\begin{aligned} & u_3 x_{0'} (\underline{x_{4'.1}\ y_{0'.1'}} + \underline{x_{4'.2}\ y_{0.2}} + \underline{x_{4'.3}\ y_{2'.1}})\ z_{0'} \\ + & u_4 x_0 (x_{4'.1}\ y_{0'.1'}\ z_0 + x_{4'.2}\ y_{0.2}\ z_0 + x_1\ y_{0'.1'}\ z_{0'} \\ & \qquad + x_2\ y_{0.2}\ z_{0'} + x_3\ y_{2'.1}\ z_{0'} + \underline{x_{4'.1}\ y_{0'.1'}\ z_{0'}} \\ & \qquad + \underline{x_{4'.2}\ y_{0.2}\ z_{0'}} + \underline{x_{4'.3}\ y_{2'.1}\ z_{0'}}). \end{aligned}$$

But each of the compound factors $x_{0'.1} y_{0'}$, $x_{0'.2} y_0$, $x_{0'.3} z_{0'}$, $x_{4'.0} z_{0'}$ is *zero*, as may be seen by application of Rule 8 or mere inspection of the Table of Limits. Hence the terms underlined vanish. Substituting the terms left, we get

$$\begin{aligned} A = u_{3'.4} (& x_{4'.1.\dot{0}}\ y_{0'.\dot{1}'}\ z_0 + x_{4'.2.\dot{0}}\ y_{\dot{2}.0}\ z_0 + x_{1.\dot{0}}\ y_{0'.1'}\ z_{0'} \\ & + x_{2.\dot{0}}\ y_{\dot{2}.0}\ z_{0'} + x_{3.\dot{0}}\ y_{2'.1}\ z_{0'}). \end{aligned}$$

But the factors *dotted underneath* in the respective terms may be omitted, for they are implied in their co-factors *of the same variable*, as

may be seen by mere inspection of the table, or by application of Rules 6 and 7. Taking, for example, the factors of the first term, we have

$$x_{1.0} = x_1\alpha + x_0\beta,$$

in which $\alpha = p\,(x_1 - x_0) = p\,(-y) = y_{0'}$,

and $\beta = p\,(x_0 - x_1) = y_0 = 0$,

because of the co-factor $y_{0'}$;

and

$$y_{0'.1'} = y_{0'}\alpha + y_{1'}\beta,$$

in which $\alpha = p'(y_0 - y_1) = p'(-\frac{1}{2}z) = z_0$,

and $\beta = p'(y_1 - y_0) = p'(\frac{1}{2}z) = z_{0'} = 0$.

The cancelling in the other terms may be similarly verified by Rules 6 and 7.

Note β.—Substituting this value of $u_{2'.4}$, we get

$$A = u_{2'.4}\,(x_{7'.4'.1}\,y_{0'} + x_{7'.4'.2}\,y_0)\,z_0 + u_{2'.4}\,(x_{7'.1}\,y_{1'} + x_{7'.2}\,y_2 + x_{7'.3}\,y_{3'.1})\,z_{0'}.$$

In the *first term* of the first bracket,

$$x_{7'.4'} = x_{7'}\alpha + x_{4'}\beta,$$

in which $\alpha = p'(x_7 - x_4) = p'\left(4a - \dfrac{y^2}{2z}\right)$
$= p\,(y^2 - 8az)$,

for $z_{0'}$ is inadmissible because of the co-factor z_0 outside the bracket, thus,

$$\alpha = p\,\{(y - \sqrt{8az})\,(y + \sqrt{8az})\}$$
$$= p'(y + \sqrt{8az}),$$

for positive values of y are inadmissible because of the co-factor $y_{0'}$;

also

$$\beta = p'(x_4 - x_7) = p'(y^2 - 8az)$$
$$= p'\,\{(y - \sqrt{8az})\,(y + \sqrt{8az})\}$$
$$= p\,(y + \sqrt{8az}).$$

Thus, $\alpha = y_{4'}$ and $\beta = y_4$.

Again, in the *second term* of the first bracket, we have

$$x_{7'.4'} = x_{7'}\alpha + x_{4'}\beta,$$

in which (as before)

$$\alpha = p\,(y^2 - 8az) = p\,\{(y - \sqrt{8az})\,(y + \sqrt{8az})\}$$
$$= p\,(y - \sqrt{8az}),$$

for *negative* values of y are inadmissible this time because of the co-factor y_0. Hence $\alpha = y_3$.

Similarly, we get $\beta = y_{3'}$.

Substituting these values of $x_{7'.4'}$ in the first and second terms respectively, we get

$$A = u_{2'.4}\{(x_{7'.1}\,y_{4'.0'} + x_{4'.1}\,y_{0'.4} + x_{7'.2}\,y_{3\,.0} + x_{4'.2}\,y_{3'.0})\,z_0 + (x_{7'.1}\,y_{1'} + x_{7'.2}\,y_2 + x_{7'.3}\,y_{2'.1})\,z_{0'}\}.$$

Applying Rule 8 to the x-statements in each term, we get

in the first term, $x_{7'.1} = x_{7'.1}\,y_6$;
in the second term, $x_{4'.1} = x_{4'.1}\,y_{5'}$,

because of the co-factors z_0 and $y_{0'}$;

in the third term, $x_{7'.2} = x_{7'.2}\,y_{3'}$;
in the fourth term, $x_{4'.2} = x_{4'.2}\,y_7$,

because of the co-factors z_0 and y_0.

Taking next the second bracket, we have

in the fifth term, $x_{7'.1} = x_{7'.1}\,y_6$;
in the sixth term, $x_{7'.2} = x_{7'.2}\,y_{3'}$;
in the seventh term, $x_{7'.3} = x_{7'.3}\,z_1$.

Substituting in every term, we get

$$A = u_{2'.4}\{(x_{7'.1}\,y_{4'.0'.6} + x_{4'.1}\,y_{0'.5'.4} + x_{7'.2}\,y_{3'.3\,.0} + x_{4'.2}\,y_{3'.0\,.7})\,z_0 + (x_{7'.1}\,y_{1'.6} + x_{7'.2}\,y_{3'.2} + x_{7'.3}\,y_{2'.1}\,z_1)\,z_{0'}\}.$$

But by inspection of the table, or by application of Rules 6 and 7, we have $y_{4'\,0'} = y_{4'}$, $y_{0'.5'} = y_{5'}$, $y_{3\;0} = y_3$, and $y_{0\,.7} = y_7$, so that the factors dotted underneath may be cancelled. Hence we get

$$A = u_{2'.4}\{(x_{7'.1}\,y_{4'.6} + x_{4'.1}\,y_{5'.4} + x_{7'.2}\,y_{3'.3} + x_{4'.2}\,y_{3'.7})\,z_0 + (x_{7'.1}\,y_{1'.6} + x_{7'.2}\,y_{3'.2} + x_{7'.3}\,y_{2'.1}\,z_1)\,z_{0'}\}.$$

Applying Rule 8 to the y-statements in each term, we get

in the *first* term, $y_{4'.6} = y_{4'.6}\,z_{2'}$;
in the *second* term, $y_{5'.4} = y_{5'.4}\,z_{2'}$;
in the *third* term, $y_{3'.3} = y_{3'.3}\,z_{3'}$;
in the *fourth* term, $y_{3'.7} = y_{3'.7}\,z_{3'}$;
in the *fifth* term, $y_{1'.6} = y_{1'.6}\,z_1$;
in the *sixth* term, $y_{3'.2} = y_{3'.2}\,z_1$;
in the *seventh* term, $y_{2'.1} = y_{2'.1}\,z_{0'}$.

Substituting in every term, we get

$$A = \{z_{2'.0}\,(y_{4'.6}\,x_{7'.1} + y_{5'.4}\,x_{4'.1}) + z_{3'.0}\,(y_{3'.3}\,x_{7'.2} + y_{3'.7}\,x_{4'.2}) + z_{0'.1}\,(y_{1'.6}\,x_{7'.1} + y_{3'.2}\,x_{7'.2} + y_{2'.1}\,x_{7'.3})\}\,u_{2'.4},$$

which, except in the arrangement of the terms, agrees with the result in the text.

On Progressive Waves. *By* Lord Rayleigh, M.A., F.R.S.

[*Read Nov.* 8, 1877.]

It has often been remarked that, when a group of waves advances into still water, the velocity of the group is less than that of the individual waves of which it is composed; the waves appear to advance through the group, dying away as they approach its anterior limit. This phenomenon was, I believe, first explained by Stokes, who regarded the group as formed by the superposition of two infinite trains of waves, of equal amplitudes and of nearly equal wave-lengths, advancing in the same direction. My attention was called to the subject about two years since by Mr. Froude, and the same explanation then occurred to me independently.* In my book on the "Theory of Sound" (§ 191), I have considered the question more generally, and have shewn that, if V be the velocity of propagation of any kind of waves whose wave-length is λ, and $\kappa = 2\pi\lambda^{-1}$, then U, the velocity of a group composed of a great number of waves, and moving into an undisturbed part of the medium, is expressed by

$$U = \frac{d(\kappa V)}{d\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

or, as we may also write it,

$$U : V = 1 + \frac{d \log V}{d \log \kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Thus, if $V \propto \lambda^n$, $\qquad U = (1-n)\, V \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$

In fact, if the two infinite trains be represented by $\cos \kappa (Vt-x)$ and $\cos \kappa'(V't-x)$, their resultant is represented by

$$\cos \kappa (Vt-x) + \cos \kappa' (V't-x),$$

which is equal to

$$2 \cos \left\{ \frac{\kappa' V' - \kappa V}{2} t - \frac{\kappa' - \kappa}{2} x \right\} . \cos \left\{ \frac{\kappa' V' + \kappa V}{2} t - \frac{\kappa' + \kappa}{2} x \right\}.$$

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* Another phenomenon, also mentioned to me by Mr. Froude, admits of a similar explanation. A steam launch moving quickly through the water is accompanied by a peculiar system of diverging waves, of which the most striking feature is the obliquity of the line containing the greatest elevations of successive waves to the wave-fronts. This wave pattern may be explained by the superposition of two (or more) infinite trains of waves, of slightly differing wave-lengths, whose directions and velocities of propagation are so related in each case that there is no change of position relatively to the boat. The mode of composition will be best understood by drawing on paper two sets of parallel and equidistant lines, subject to the above condition, to represent the crests of the component trains. In the case of two trains of slightly different wave-lengths, it may be proved that the tangent of the angle between the line of maxima and the wave-fronts is half the tangent of the angle between the wave-fronts and the boat's course.

If $\kappa' - \kappa$, $V' - V$ be small, we have a train of waves whose amplitude varies slowly from one point to another between the limits 0 and 2, forming a series of groups separated from one another by regions comparatively free from disturbance. The position at time t of the middle of that group, which was initially at the origin, is given by

$$(\kappa' V' - \kappa V)\, t - (\kappa' - \kappa)\, x = 0,$$

which shews that the velocity of the group is $(\kappa' V' - \kappa V) \div (\kappa' - \kappa)$. In the limit, when the number of waves in each group is indefinitely great, this result coincides with (1).

The following particular cases are worth notice, and are here tabulated for convenience of comparison :—

$V \propto \lambda$,	$U = 0$,	Reynolds' disconnected pendulums.
$V \propto \lambda^{\frac{1}{2}}$,	$U = \frac{1}{2} V$,	Deep-water gravity waves.
$V \propto \lambda^{0}$,	$U = V$,	Aërial waves, &c.
$V \propto \lambda^{-\frac{1}{2}}$,	$U = \frac{3}{2} V$,	Capillary water waves.
$V \propto \lambda^{-1}$,	$U = 2V$,	Flexural waves.

The capillary water waves are those whose wave-length is so small that the force of restitution due to capillarity largely exceeds that due to gravity. Their theory has been given by Thomson (Phil. Mag., Nov. 1871). The flexural waves, for which $U = 2V$, are those corresponding to the bending of an elastic rod or plate ("Theory of Sound," § 191).

In a paper read at the Plymouth meeting of the British Association (afterwards printed in "Nature," Aug. 23, 1877), Prof. Osborne Reynolds gave a dynamical explanation of the fact that a group of deep-water waves advances with only half the rapidity of the individual waves. It appears that the energy propagated across any point, when a train of waves is passing, is only one-half of the energy necessary to supply the waves which pass in the same time, so that, if the train of waves be limited, it is impossible that its front can be propagated with the full velocity of the waves, because this would imply the acquisition of more energy than can in fact be supplied. Prof. Reynolds did not contemplate the cases where *more* energy is propagated than corresponds to the waves passing in the same time; but his argument, applied conversely to the results already given, shews that such cases must exist. The ratio of the energy propagated to that of the passing waves is $U : V$; thus the energy propagated in the unit time is $U : V$ of that existing in a length V, or U times that existing in the unit length. Accordingly

Energy propagated in unit time : Energy contained (on an average) in unit length $= d(\kappa V) : d\kappa$, by (1).

As an example, I will take the case of small irrotational waves in water of finite depth l.* If z be measured downwards from the surface, and the elevation (h) of the wave be denoted by

$$h = H\cos(nt - \kappa x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

in which $n = \kappa V$, the corresponding velocity-potential (ϕ) is

$$\phi = -VH\frac{e^{\kappa(z-l)} + e^{-\kappa(z-l)}}{e^{\kappa l} - e^{-\kappa l}}\sin(nt - \kappa x) \quad \ldots\ldots\ldots\ldots (5).$$

This value of ϕ satisfies the general differential equation for irrotational motion ($\nabla^2\phi = 0$), makes the vertical velocity $\frac{d\phi}{dz}$ zero when $z = l$, and $-\frac{dh}{dt}$ when $z = 0$. The velocity of propagation is given by

$$V^2 = \frac{g}{\kappa}\,\frac{e^{\kappa l} - e^{-\kappa l}}{e^{\kappa l} + e^{-\kappa l}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

We may now calculate the energy contained in a length x, which is supposed to include so great a number of waves that fractional parts may be left out of account.

For the potential energy we have

$$V_1 = g\rho\iint_0^h z\,dz\,dx = \tfrac{1}{2}g\rho\int h^2\,dx = \tfrac{1}{4}g\rho H^2 . x \quad \ldots\ldots\ldots\ldots (7).$$

For the kinetic energy,

$$T = \tfrac{1}{2}\rho\iint\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\}dx\,dz$$

$$= \tfrac{1}{2}\rho\int\left(\phi\frac{d\phi}{dz}\right)_{z=0}dx = \tfrac{1}{4}g\rho H^2 . x \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

by (1) and (6). If, in accordance with the argument advanced at the end of this paper, the equality of V_1 and T be assumed, the value of the velocity of propagation follows from the present expressions. The whole energy in the waves occupying a length x is therefore (for each unit of breadth) $V_1 + T = \tfrac{1}{2}g\rho H^2 . x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$,

H denoting the maximum elevation.

We have next to calculate the energy propagated in time t across a plane for which x is constant, or, in other words, the work (W) that must be done in order to sustain the motion of the plane (considered as a flexible lamina) in the face of the fluid pressures acting upon the front of it. The variable part of the pressure (δp), at depth z, is given by

$$\delta p = -\rho\frac{d\phi}{dt} = -n\rho VH\frac{e^{\kappa(z-l)} + e^{-\kappa(z-l)}}{e^{\kappa l} - e^{-\kappa l}}\cos(nt - \kappa x),$$

* Prof. Reynolds considers the trochoidal wave of Rankine and Froude, which involves molecular rotation.

while for the horizontal velocity

$$\frac{d\phi}{dx} = \kappa VH \frac{e^{\kappa(z-l)} + e^{-\kappa(z-l)}}{e^{\kappa l} - e^{-\kappa l}} \cos(nt - \kappa x);$$

so that $$W = \iint \delta p \frac{d\phi}{dx} dz\, dt = \tfrac{1}{4} g\rho H^2 . Vt . \left[1 + \frac{4\kappa l}{e^{2\kappa l} - e^{-2\kappa l}}\right] \quad \ldots\ldots (10),$$

on integration. From the value of V in (6) it may be proved that

$$\frac{d(\kappa V)}{d\kappa} = \tfrac{1}{2} V \left\{1 + \frac{1}{V^2} \frac{d(\kappa V^2)}{d\kappa}\right\} = \tfrac{1}{2} V \left\{1 + \frac{4\kappa l}{e^{2\kappa l} - e^{-2\kappa l}}\right\};$$

and it is thus verified that the value of W for a unit time

$$= \frac{d(\kappa V)}{d\kappa} \times \text{energy in unit length.}$$

As an example of the direct calculation of U, we may take the case of waves moving under the joint influence of gravity and cohesion.

It is proved by Thomson that

$$V^2 = \frac{g}{\kappa} + T'\kappa \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where T' is the cohesive tension. Hence

$$U = \tfrac{1}{2} V \left\{1 + \frac{1}{V^2} \frac{d(\kappa V^2)}{d\kappa}\right\} = \tfrac{1}{2} V \frac{g + 3\kappa^2 T}{g + \kappa^2 T} \quad \ldots\ldots\ldots (12).$$

When κ is small, the surface tension is negligible, and then $U = \frac{1}{2} V$; but when, on the contrary, κ is large, $U = \frac{3}{2} V$, as has already been stated. When $T\kappa^2 = g$, $U = V$. This corresponds to the minimum velocity of propagation investigated by Thomson.

Although the argument from interference groups seems satisfactory, an independent investigation is desirable of the relation between energy existing and energy propagated. For some time I was at a loss for a method applicable to all kinds of waves, not seeing in particular why the comparison of energies should introduce the consideration of a variation of wave-length. The following investigation, in which the increment of wave-length is *imaginary*, may perhaps be considered to meet the want:—

Let us suppose that the motion of every part of the medium is resisted by a force of very small magnitude proportional to the mass and to the velocity of the part, the effect of which will be that waves generated at the origin gradually die away as x increases. The motion, which in the absence of friction would be represented by $\cos(nt - \kappa x)$, under the influence of friction is represented by $e^{-\mu x} \cos(nt - \kappa x)$, where μ is a small positive coefficient. In strictness the value of κ is also altered by the friction; but the alteration is of the second order as regards the frictional forces, and may be omitted under the circumstances here supposed. The energy of the waves per unit length at

any stage of degradation is proportional to the square of the amplitude, and thus the whole energy on the positive side of the origin is to the energy of so much of the waves at their greatest value, *i. e.*, at the origin, as would be contained in the unit of length, as $\int_0^\infty e^{-2\mu x}\,dx : 1$, or as $(2\mu)^{-1} : 1$. The energy transmitted through the origin in the unit time is the same as the energy dissipated; and, if the frictional force acting on the element of mass m be hmv, where v is the velocity of the element and h is constant, the energy dissipated in unit time is $h\Sigma mv^2$ or $2hT$, T being the kinetic energy. Thus, on the assumption that the kinetic energy is half the whole energy, we find that the energy transmitted in the unit time is to the greatest energy existing in the unit length as $h : 2\mu$. It remains to find the connection between h and μ.

For this purpose it will be convenient to regard $\cos(nt-\kappa x)$ as the real part of $e^{int}e^{-i\kappa x}$, and to inquire how κ is affected, when n is given, by the introduction of friction. Now the effect of friction is represented in the differential equations of motion by the substitution of $\frac{d^2}{dt^2}+h\frac{d}{dt}$ in place of $\frac{d^2}{dt^2}$, or, since the whole motion is proportional to e^{int}, by substituting $-n^2+ihn$ for $-n^2$. Hence the introduction of friction corresponds to an alteration of n from n to $n-\frac{1}{2}ih$ (the square of h being neglected); and accordingly κ is altered from κ to $\kappa-\frac{1}{2}ih\frac{d\kappa}{dn}$. The solution thus becomes $e^{-\frac{1}{2}h\frac{d\kappa}{dn}x}e^{i(nt-\kappa x)}$, or, when the imaginary part is rejected, $e^{-\frac{1}{2}h\frac{d\kappa}{dn}x}\cos(nt-\kappa x)$; so that $\mu=\frac{1}{2}h\frac{d\kappa}{dn}$, and $h : 2\mu = \frac{dn}{d\kappa}$. The ratio of the energy transmitted in the unit time to the energy existing in the unit length is therefore expressed by $\frac{dn}{d\kappa}$ or $\frac{d(\kappa V)}{d\kappa}$, as was to be proved.

It has often been noticed, in particular cases of progressive waves, that the potential and kinetic energies are equal; but I do not call to mind any general treatment of the question. The theorem is not usually true for the individual parts of the medium,* but must be understood to refer either to an integral number of wave-lengths, or to a space so considerable that the outstanding fractional parts of waves may be left out of account. As an example well adapted to give insight into the question, I will take the case of a uniform stretched circular membrane ("Theory of Sound," § 200) vibrating with a given

* Aërial waves are an important exception.

number of nodal circles and diameters. The fundamental modes are not quite determinate in consequence of the symmetry, for any diameter may be made nodal. In order to get rid of this indeterminateness, we may suppo e the membrane to carry a small load attached to it anywhere except on a nodal circle. There are then two definite fundamental modes, in one of which the load lies on a nodal diameter, thus producing no effect, and in the other midway between nodal diameters, where it produces a maximum effect ("Theory of Sound," § 208). If vibrations of both modes are going on simultaneously, the potential and kinetic energies of the whole motion may be calculated by *simple addition* of those of the components. Let us now, supposing the load to diminish without limit, imagine that the vibrations are of equal amplitude and differ in phase by a quarter of a period. The result is a *progressive* wave, whose potential and kinetic energies are the sums of those of the stationary waves of which it is composed. For the first component we have $V_1 = E\cos^2 nt$, $T_1 = E\sin^2 nt$; and for the second component, $V_2 = E\sin^2 nt$, $T_2 = E\cos^2 nt$; so that $V_1 + V_2 = T_1 + T_2 = E$, or the potential and kinetic energies of the progressive wave are equal, being the same as the whole energy of either of the components. The method of proof here employed appears to be sufficiently general, though it is rather difficult to express it in language which is appropriate to all kinds of waves.

Notes on Vortex-Motion, on the Triple Generation of Three-bar Curves, and on the Mass-Centre of an Octahedron. By Prof. Clifford, F.R.S.

[*Read November* 8, 1877.]

On Vortex-Motion.

Let σ be the velocity, and ω the rotation, at any point of a moving substance. It is known that $2\omega = V\nabla\sigma$; viz., this is equivalent to the three equations ordinarily written thus:

$$2\xi = \delta_z v - \delta_y w,$$
$$2\eta = \delta_x w - \delta_z u,$$
$$2\zeta = \delta_y u - \delta_x v.$$

If, moreover, k be the *expansion*, or the logarithmic rate of change of the volume, we have $k = -S\nabla\sigma$; viz., this is $\delta_x u + \delta_y v + \delta_z w$. Hence the quaternion q, $= -k + 2\omega$, is simply $\nabla\sigma$. The problem solved by

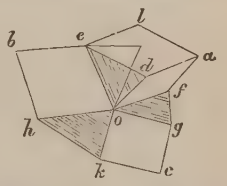
Stokes as a general question of analysis, and subsequently by Helmholtz for the special case of fluid motion, may be stated as follows:—Given the expansion and the rotation at every point of a moving substance, it is required to find the velocity at every point. In symbols, it being known that $q = \nabla\sigma$, and q being given, it is required to find σ.

The solution of the problem is exhibited in a very simple form if we consider it a little more generally. A quaternion q is given at every point of space; it is required to find a quaternion r so that $q = \nabla r$. The solution is that $\nabla q = \nabla^2 r$, and therefore r is the potential of ∇q; that is, $r_a = \int \frac{\nabla q_b \,.\, dv_b}{D_{ab}}$, where r_a means the value of r at the point a, δv_b means an element of volume at the point b, and D_{ab} the distance between the two points a b. If r is a pure vector, so that $Sr = 0$, q must satisfy a certain condition, namely, we must have $S\nabla q = 0$ everywhere. This is identically satisfied if $q = -k + 2\omega$, where ω is the rotation at any point of a moving substance; for

$$2S\nabla\omega = S\nabla^2\sigma = 0.$$

On the Triple Generation of Three-bar Curves.

The theorem on which the triple generation of three-bar curves depends has been stated as follows by Prof. Cayley:—Let the triangles *deo*, *fog*, *ohk* be similar, and the figures *adof*, *cgok*, *bhoe* parallelograms. Then the triangle *abc* will be similar to *deo*, &c.



The proof of this is intuitive if we consider the operation which converts *oh* into *ok*. This operation consists in turning through the angle *hok*, and altering the length in the ratio *oh* : *ok*. The same operation converts *eb* into *gc*, *de* into *do*, and therefore into *af*, and *ad* or *fo* into *fg*. Hence *ad*, *de*, *eb* are converted by this operation into *fg*, *af*, *gc*, and therefore the whole line *ab* is converted by it into the whole line *ac*. That is to say, the triangle *abc* is similar to *ohk*, as was to be proved.

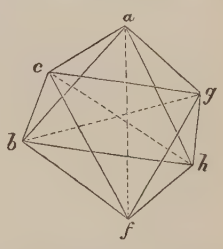
If we complete the parallelogram *adel*, this amounts to saying that the broken lines *aleb*, *afgc* are similar to one another.

If one of the three-bar systems is a crossed rhomboid, the other two are kites. This would of course follow from the known fact that the path of the moving point in both these cases is the inverse of a conic. But

it is also intuitively obvious as soon as the figure is drawn, and thus supplies an elementary proof that the path *is* the inverse of a conic in the case of a kite, which is not otherwise easy to get.

On the Mass-Centre of an Octahedron.

Let *af*, *bg*, *ch* be three finite lines not meeting. By an *octahedron* I mean the solid whose eight faces are *abc*, *acg*, *agh*, *ahb*, *fbc*, *fcg*, *fgh*, *fhb*. If this solid figure be filled with matter of uniform density, its mass-centre may be found by a very simple construction.



The solid is girdled by three skew quadrilaterals *bcgh*, *cahf*, *abfg*. Now the middle points of the sides of any skew quadrilateral are in one plane. Draw, then, three planes bisecting the sides of these quadrilaterals, and let them meet in a point k; which, following Sylvester in a paper to be presently mentioned, I will call the *cross-centre*. Let also m be the mean point of the six vertices a, b, c, f, g, h; it is the mass-centre of the triangle formed by the middle points of *af*, *bg*, *ch*. To find s, the mass-centre of the solid, join km and produce it to s so that $ms = \frac{1}{2}km$.

The proof is that the solid is the sum of the four tetrahedra *afbc*, *afcg*, *afgh*, *afhb*. Now the mass-centre of a tetrahedron is the mean point of its vertices; consequently the line joining the mass-centre of *afbc* to the middle point of *gh* is divided by the point m in the ratio 1 : 2. The same is true of the other three tetrahedra and the middle points of *hb*, *bc*, *cg*. Hence the mass-centres of the four tetrahedra are in one plane passing through the point s found by the above construction, and therefore the mass-centre of the whole solid is in this plane. So also it is in the other two planes determined by dividing the solid into tetrahedra having the common edge *bg* and the common edge *ch* respectively. Therefore it coincides with the point s.

This remark was suggested to me by Sylvester's construction for the mass-centre of a tetrahedral frustum, of which it is a simple extension. In fact, by making the pairs of faces *abh*, *ahg*; *acg*, *cfg*; *cbf*, *bhf* to be respectively coplanar, we pass at once to that particular case.

Prof. Cayley read a paper "On the function $\phi x = \frac{ax+b}{cx+d}$ in connexion with the matrix $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$." The value obtained for the n-th function is substantially of the same form as that found long ago by Babbage, but is more compendiously expressed; viz., the result is

$$\phi^n(x) = \frac{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(\quad cx-a)},$$

where

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(a+d)^2}{ad-bc}.$$

It was arrived at in a simple manner by means of the identity

$$\begin{pmatrix} M-a, & b \\ c, & M-d \end{pmatrix} = 0, \quad \text{or} \quad M^2-(a+d)M+(ad-bc) = 0,$$

satisfied by the quadric matrix $M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$.

Prof. Cayley gave also an account of researches on which he is engaged upon the double ϑ-functions. In view to these he establishes in a strictly analogous manner the theory of the single ϑ-functions, the process for the single functions being in fact as follows:—Considering u, x as connected by the differential relation

$$\delta u = \frac{\delta x}{\sqrt{a-x \,.\, b-x \,.\, c-x \,.\, d-x}},$$

then, if A, B, C, D, Ω denote functions of u [viz., for shortness, the single letters are used, instead of writing them as functional symbols, $A(u)$, $B(u)$, &c.], then, by way of definition of these functions (called, the first four of them ϑ-functions, and the last an ω-function), we assume

$A, B, C, D = \Omega\sqrt{a-x}, \ \Omega\sqrt{b-x}, \ \Omega\sqrt{c-x}, \ \Omega\sqrt{d-x}$ respectively,

together with one other equation, as presently mentioned. Without in any wise defining the meaning of Ω, we then obtain a set of equations of the form

$$A\delta B - B\delta A = \Omega^2\sqrt{c-x \,.\, d-x}\,\delta u$$

(mere constant coefficients are omitted), or, what is the same,

$$A\delta B - B\delta A = CD,$$

which are differential equations defining the nature of the ratio-functions $A : B : C : D$. If, proceeding to second differential coefficients, we attempt to form the expressions for $A\delta^2 A - (\delta A)^2$, &c., these involve multiples of $\Omega\delta^2\Omega - (\delta\Omega)^2$, and in order to obtain a convenient

form we assume $\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M (\delta u)^2$, where M is a function of x. We thus obtain an equation $A\delta^2 A - (\delta A)^2 = \Omega^2 \mathfrak{A} (\delta u)^2$, where the value of $\mathfrak{A}$ depends upon that of M. The value of M has to be taken so as to simplify as much as may be the expression of $\mathfrak{A}$, but so that M shall be a symmetrical function of the constants a, b, c, d [this last condition in order that the like simplification may present itself in the analogous relations $B\delta^2 B - (\delta B)^2 = \Omega^2 \mathfrak{B} (\delta u)^2$, &c.]. The proper expression of M is found to be

$$M = -2x^2 + x(a+b+c+d) + a^2 + b^2 + c^2 + d^2 - 2bc - 2ca - 2ab - 2ad - 2bd - 2cd,$$

viz., M having this value, the one other equation above referred to is

$$\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M (\delta u)^2;$$

and we then have a system of four equations such as

$$A\delta^2 A - (\delta A)^2 = \Omega^2 \mathfrak{A} (\delta u)^2,$$

where $\mathfrak{A}$ is a linear function of x, and where consequently $\Omega^2\mathfrak{A}$ can be expressed as a linear function of any two of the four squares A^2, B^2, C^2, D^2.

To establish the connexion with the Jacobian H and Θ functions, the differential relation between u, x may be taken to be

$$\delta u = \frac{\delta x}{\sqrt{x \,.\, 1-x \,.\, 1-k^2 x}};$$

viz., we have here $d = \infty$, and to adapt the formulæ to this value it is necessary to write $\frac{u}{\sqrt{d}}$ instead of u, and make other easy changes. The result is that Ω differs from D by a constant factor only, so that, instead of the five functions A, B, C, D, Ω, we have only the four functions A, B, C, D. The equation $\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M (\delta u)^2$ is replaced by an equation $D\delta^2 D - (\delta D)^2 = \mathfrak{D} (\delta u)^2$, or say $\delta^2 (\log D) = D^2 \mathfrak{D} (\delta u)^2$, which gives a result of the form

$$D = e^{u + \lambda^2 \int \delta u \int \delta u \frac{A^2}{D^2}},$$

showing that D differs from Jacobi's $\Theta(u)$ only by an exponential factor of the form $Ce^{\lambda u^2}$; and it then further appears that A, B, C differ only by factors of the like form from the three numerator functions $\mathrm{H}(u)$, $\mathrm{H}(u+\kappa)$, $\Theta(u+\kappa)$, so that, neglecting constant factors, the functions

$$\frac{A}{D}, \frac{B}{D}, \frac{C}{D} \quad \text{are equal to} \quad \frac{\mathrm{H}(u)}{\Theta(u)}, \frac{\mathrm{H}(u+\kappa)}{\Theta(u)}, \frac{\Theta(u+\kappa)}{\Theta(u)};$$

that is, to the elliptic functions $\mathrm{sn}\, u$, $\mathrm{cn}\, u$, $\mathrm{dn}\, u$.

December 13th, 1877.

C. W. MERRIFIELD, Esq., V.P., F.R.S., in the Chair.

The Rev. W. Ellis, B.A., was elected a Member; and Mr. F. B. W. Phillips, B.A., was proposed for election.

The following communications were made to the Society:—

"Notes on Normals:" Mr. S. Roberts.

"On the Geometrical Representation of Imaginary Quantities and the real (m, n) Correspondence of two Planes:" Prof. Cayley.

Mr. Perigal (the Auditor) stated that he had examined the Treasurer's accounts, and had found them correct.

The following presents were made to the Society:—

"Teoremi stereometrici dai quali si deducono le proprietà dell' esagramme di Pascal," di L. Cremona (Roma, 1877): from the Author.

"Proceedings of the Royal Society of Edinburgh," Session 1862-3—1876-7, Vol. v. 1—4, vi. 1—3, vii. 1—3, viii. 1—3, ix. 1—No. 98.

"Bulletin des Sciences Mathématiques et Astronomiques," Deuxième Série, Tome i., Sept. 1877.

"Monatsbericht," Sept. and Oct. 1877.

"Bulletin de la Société Mathématique de France," Tome vi., No. 1, Paris, 1878.

On the Geometrical Representation of Imaginary Variables by a Real Correspondence of two Planes. By Prof. Cayley.

[*Read December 13th*, 1877.]

In my recently published paper, "Geometrical Illustration of a Theorem relating to an Irrational Function of an Imaginary Variable" [Proc. L. M. S., t. viii. (1877), pp. 212—214], I remark as follows:—"If we have v a function of u determined by an equation $f(u, v) = 0$, then to any given imaginary value $x+iy$ of u there belong two or more values, in general imaginary, of v; and for the complete understanding of the relation between the two imaginary variables we require to know the series of values $x'+iy'$ which correspond to a given series of values $x+iy$ of v, u respectively. We must, for this purpose, take x, y as the coordinates of a point P in a plane Π, and x', y' as the coordinates of a corresponding point P' in another plane Π';"—and I then proceed to consider the particular case where the equation between u, v is $u^2+v^2=a^2$, that is, where $(x+iy)^2+(x'+iy')^2=a^2$.

The general case is that of an equation $(*)\ (u, 1)^m (v, 1)^n = 0$, where to each given value, real or imaginary, of u, there correspond n

real or imaginary values of v; and to each given value, real or imaginary, of v, there correspond m real or imaginary values of u. And then, writing $u = x+iy$ and $v = x'+iy'$, and regarding (x, y), (x', y') as the coordinates of the points P, P' in the two planes Π, Π' respectively, we have a *real* (m, n) correspondence between the two planes; viz., to each real point P in the first plane there correspond n real points P' in the second plane, and to each real point P' in the second plane there correspond m real points P in the first plane. But such real correspondence of two planes does not of necessity arise from an equation between the two imaginary variables u, v; and the question of the real correspondence of two planes may be considered in itself, without any reference to such origin.

I was under the impression that the theory was a known one; but I have not found it anywhere set out in detail. It is to be noticed that, although intimately connected with, it is quite distinct from (and seems to me to go beyond) that of a Riemann's surface. Riemann represents the value u, $= x+iy$, by a point P whose coordinates are x, y; but he considers u', $= x'+iy'$, as a given imaginary value attached to the point P, without representing this value by a point P', coordinates x', y'.

I proceed to consider the general theory of the real (m, n) correspondence. Points in the first plane are denoted by the unaccented letters P, Q ...; and the corresponding points in the second plane are in general denoted by the *same* letters accented; but there are, as will be explained, special points V, W where the letters are interchanged; viz., to the points V or W in the first plane correspond points W' or V' in the second plane.

1. To a point P there correspond in general n distinct points P'; and as P varies continuously, each of the points P' also varies continuously.

2. There are certain points V called branch-points (Verzweigungspuncte), such that to each point V there correspond two united points, represented by (W'), and $n-2$ other distinct points W'. The points (W') are called cross-points, and the number of them is of course equal to that of the branch-points V.

It is throughout assumed that a point denoted by a letter other than V is not a point V.

3. If the point P, moving continuously, describe a closed curve so as to return to its original position, then, if this curve includes within it no point V (or all the points V),* each of the corresponding points P'

* The two cases of the closed curve including no point V, and including all the points V, are really identical, as the discontinuity at infinity may be disregarded. It is to be observed that, this being so, it follows that the number of the points V must be even.

will describe continuously a closed curve returning into its original position. Supposing that the curve described by P is an oval (non-autotomic closed curve), and taking this to be in the first instance an indefinitely small oval, then the curves described by the points P' will in the first instance be each of them an indefinitely small oval; but it is worth while to notice how, as the oval described by P increases, any one of the ovals described by a point P' may become autotomic; viz., if the oval described by P passes through two points Q, Q of the m points Q which correspond in the first plane to the same Q' in the second plane, then Q' will be a node in the closed curve described by that point P' which in the course of its motion comes to pass through Q'. This curve is in general an inloop curve composed of two loops, one wholly within the other (united at the point Q'), and such that they each include one and the same point V' (viz., V' is included within the inner loop): as to this, see *post*, Nos. 9 and 10. It will be observed that this node Q' is not a point (W') nor any other special point of the second plane.

4. Consider, as before, P as describing a closed curve which does not include within it any point V, and the corresponding points P' as describing each of them a closed curve. As the curve described by P approaches a point V, the curves described by two of the points P' will approach the corresponding point (W'); and when the curve described by P passes through V, the curves described by the two points P' will unite together at this point (W') as a node; viz., they will form a figure of eight,* the crossing being at the cross-point (W'), which corresponds to the branch-point V. And, corresponding to the closed curve described by P, we have this figure of eight (replacing two of the original n closed curves), and $n-2$ closed curves described by the other points P'.

5. Supposing, next, that the closed curve described by P (instead of passing through the point V) includes within it the point V, then the figure of eight transforms itself into a twice-indented oval.* There are on this curve two of the points P' which correspond to the given point P; and as P, moving continuously in its closed curve, returns to its original position, the first of these points P', moving continuously along a portion of the curve, comes to coincide with the original position of the second point P'; while the second point P', moving continuously

* The name *figure of eight* refers to the case where the two curves which come to unite at (W') are proper ovals (non-autotomic closed curves). They might have one or both of them a node or nodes, as explained in No. 3, and the term would then be inappropriate. And so, lower down, the name *twice-indented oval* is used to express the form into which a proper figure of eight is changed by the disappearance of the node.

along the remaining portion of the curve, comes to coincide with the original position of the first point P'; viz., the two portions of the curve are described by the two points P' respectively. The curve may thus be regarded as a bifid curve, belonging to these two points P'. And, corresponding to the closed curve described by P, we have this bifid curve belonging to the two points P', and $n-2$ single closed curves belonging to the other $n-2$ points P' respectively.

6. If the closed curve described by P (including within it a point V) comes to pass through a second point V, the effect will be a new node at the corresponding point (W'); viz., at this point (W') either the bifid curve unites itself with one of the single curves, or two of the single curves unite together, or the bifid curve there cuts itself. And then, if the curve described by P comes to include within it this second point V, then in the three cases respectively:—the bifid curve takes to itself the single curve, so that the system then is a trifid curve and $n-3$ single curves; or the two single curves give rise to a bifid curve, so that the system is two bifid curves and $n-4$ single curves; or, lastly, the bifid curve breaks up into two single curves, so that the system resumes its original form of n single curves.

7. We thus see how the closed curve described by P, including within it certain of the points V, may be such as to have corresponding to it an α-fid curve, a β-fid curve, &c. $(\alpha+\beta+\ldots=n)$; viz., an α-fid curve contains upon it α of the points P' which correspond to the original position of P; and then, as P describes continuously its closed curve, returning to its original position, each of these points P' describes a portion of the α-fid curve, passing from its original position to the original position of a point P' next to it upon the α-fid curve; and the like as to a β-fid curve, &c. The numbers $\alpha, \beta, \ldots$ are not of necessity unequal, and we may have sets of equal numbers in any manner. It is hardly necessary to remark that, if the curve described by P passes through any point or points V, then two of the curves described by the points P' will unite together, or it may be one of these will cut itself at the corresponding point or points (W'); and further that, as in No. 3, if the curve described by P passes through two or more of the points Q which correspond to the same point Q', then any such point Q' will present itself as a node upon the curve belonging to some point, or set of points, P'. But the order of succession in which the original n single curves unite themselves together into multifid curves, or again break up into single curves, cannot, it would appear, be explained in any general manner, and would in each case depend on the nature of the particular correspondence.

8. We may consider the case where the closed curve described by P

cuts itself. The curve may here be considered as made up of two or more ovals, or, to use a more appropriate term, say loops, each such loop being a curve not cutting itself; and the case is thus reducible to that before considered, where the curve does not cut itself. Thus, to fix the ideas, let the curve be a figure of eight, the initial position of P being at the crossing, and let neither of the loops contain within it a point V. Then, as P passes continuously along one of the loops, returning to its original position, each of the corresponding points P' describes a closed curve, which will be in the nature of a loop, viz., the initial and final directions of the motion of P not being continuous with each other, the initial and final directions of the motions of each point P' will not be continuous with each other, or there will be at the point P' an abrupt change in the direction of the curve. Similarly, as P describes the other loop of the figure of eight, each of the points P' will describe another loop; and the two loops belonging to the same point P' will unite together so as to form a figure of eight; viz., to the figure of eight described by P there will correspond figures of eight described by the n points P' respectively.

9. But consider next the case where the two loops of the curve described by P include each of them one and the same point V. This implies that one of the two loops lies inside the other, or that the curve is what has been called an inloop curve. As P, which is in the first instance taken to be at the node, passing continuously along one of the loops, returns to its original position, there are two of the points P' such that the first of these passes from its original position to the original position of the second, and the second of them passes from its original position to the original position of the first of them. We have thus two arcs between these two points P'; but inasmuch as the initial and final directions of motion of the point P are not continuous with each other, these two arcs are not continuous in direction at the two points P', but at each of these points P' the two arcs meet at an angle. As P describes the other loop, we have in like manner two arcs between the same two points P', these arcs at each of the points P' meeting at an angle; but they join on to the first-mentioned two arcs in such manner as to form two ovals intersecting each other in the two points P'. Corresponding to the inloop curve described by P, we have this pair of intersecting ovals, described by two of the points P', and $n-2$ other curves described by the other points P', and being each of them (I assume) an inloop curve.

10. If we attend only to one of the two intersecting ovals, we have in the first plane an inloop curve, and corresponding thereto in the second plane an oval passing through two of the points P' which correspond to the node P of the inloop curve. Interchanging the two planes,

and writing Q instead of P, we have in the first plane an oval passing through two of the points Q which correspond to a point Q'; and corresponding to this oval we have in the second plane an inloop curve having this point Q for its node, viz., these are the corresponding figures mentioned in No. 3.

11. Consider a given point Q; and let the corresponding points Q' be called (selecting the suffixes at pleasure) $Q'_1, Q'_2 \ldots Q'_n$. Taking then a point O indefinitely near to Q, the corresponding points O' will be indefinitely near to $Q'_1, Q'_2 \ldots Q'_n$ respectively, and they will be called $O'_1, O'_2 \ldots O'_n$ accordingly. It is to be observed that by the indefinitely near point O is meant a point such that the distance from O to Q is indefinitely small in comparison with the distance of either of these points from any point V; so that we cannot have from Q to O two indefinitely short paths including between them a point V; or say so that the indefinitely short path from Q to O is determinate.

Proceeding in this manner from Q to O, and so through a succession of indefinitely near points to a distant point S, we seem to determine the suffixes of the corresponding points S'; but, by what precedes, it appears that such determination for a point S is dependent on the path from Q to S; and consequently that we do not thus obtain a proper determination of the suffixes of the points S'. In fact, if we were to pass from Q by a path including one or more of the points V back to Q, we should obtain for the several points Q' respectively suffixes which are in general different from the suffixes originally given to these points respectively.

12. The difficulty is got over as follows:—Considering as before the given point Q, and calling the corresponding points $Q'_1, Q'_2 \ldots Q'_n$ at pleasure, we pass from Q to the indefinitely near point O, and thence, by so many paths chosen at pleasure, to the several branch-points V; these paths from O to the several points V are called *barriers*. To fix the ideas, we may consider these as non-autotomic non-intersecting lines drawn from O to the several points V. Consider the barrier from O to one of these points V; as P passes along this barrier from O to V, two of the corresponding points P' will pass from two of the corresponding points O' to the corresponding cross-point (W'); the paths of these two points is called the *counter-barrier* corresponding to the barrier in question; and we have thus in the second plane a system of counterbarriers, each drawn from two points O' to meet in a point (W'). By what precedes, the points O' have each of them a determinate suffix; a counterbarrier is thus drawn from two points with given suffixes, suppose O'_1 and O'_2, to a point (W'), and this may be distinguished accordingly as a counterbarrier 12; and in like manner the cross-point (W') through which it passes will be called a cross-point (W'_{12}); and

the barrier corresponding hereto, and the branch-point V at which it terminates, will in like manner be called a barrier 12, and a branch-point V_{12}. Each barrier and branch-point will thus have a pair of suffixes; and the corresponding counterbarrier and cross-point will have the same pair of suffixes. It is to be observed that two or more of these corresponding figures may very well have the same pair of suffixes; but that such corresponding figures must be distinguished from each other; thus, if there are two branch-points V_{12}, these may be distinguished as the branch-points αV_{12} and βV_{12}, and the barriers, counterbarriers, and cross-points by means of these same letters α and β, (or otherwise,) as may be convenient. It would seem that not only the number of the points V must be even, but the number of each set of points V_{12} must also be even (see *post*, No. 15).

13. It is also to be noticed that the determination of the suffixes of the several points V, &c. depends *first* upon the arbitrary choice of the suffixes of the points Q', and *next* on the choice of the system of barriers; but that, *these being assumed*, the suffixes of the several points V, &c. are completely determinate.

14. Taking now any point S whatever, and supposing that P moves from Q continuously to S by a path which does not meet a barrier, the points P' will move from the several points Q' to the several points S' by paths not meeting the counterbarriers; viz., to each point S' there will be a path from some point Q'; and giving to such point S' the suffix of the point Q', the suffixes of the several points S' which correspond to any point whatever, S, will be completely determined. The determination depends of course on the assumptions referred to No. 13, but not in anywise on the position of the point S.

It will be noticed that, as all the points V are connected together by the barriers, the only closed paths from a point to itself are paths not including any, or including all, of the points V; and that between such paths there is no real distinction.

15. Consider a point P moving continuously in any manner. The several corresponding points $P'_1, P'_2 \ldots P'_n$ will each of them move continuously, but the suffixes interchange; viz., when P arrives at and then passes over a barrier $\alpha\beta$, the corresponding points P'_α and P'_β will each arrive at the corresponding counterbarrier $\alpha\beta$, and, on passing over this, P'_α will be changed into P'_β and P'_β into P'_α, the other points P' remaining unchanged; and the like in other cases. This in fact includes the whole or the greater part of the foregoing theory. Thus, if P describe a closed curve not cutting any barrier, there will be no change of suffix, and when P returns to its original position each of the corresponding points $P'_1, P'_2 \ldots P'_n$ will describe a closed curve, return-

ing to its original position. But suppose that P describes a closed curve, cutting once only a barrier 12; suppose that the path is from P to Q, and then crossing the barrier to R, and thence again to P; P_1' passes to Q_1', and then crossing the counterbarrier it passes from R_2' to P_2'; while at the same time P_2' passes to Q_2', and then crossing the counterbarrier it passes from R_1' to P_1'; viz., we have P_1', P_2' describing the two portions of a bifid curve. If there were only a single branch-point V_{12}, and therefore only a single barrier OV_{12}, then we might have through P a closed curve cutting OV_{12} once only, and including within it the point O, but not including within it the point V_{12}; and here there ought not to be a bifid curve, but the points P_1', P_2' ought to describe each of them a single curve. But supposing there are two points V_{12}, and consequently two barriers OV_{12} (meeting in O), then the closed curve, meeting once only a barrier 12 (viz., it meets only one such barrier, and that once only), must include within one and only one of the two points V_{12}; and in this case there ought to be a bifid curve. It is by such reasoning as this that I infer the foregoing theorem (No. 12), that the number of each set of points V_{12} is even.

16. We may consider how the suffixes are affected when, instead of the original system of barriers, we have a new system of barriers. I suppose that we have in the two cases respectively the same point Q, and the same suffixes for the points $Q_1', Q_2' \ldots Q_n'$ which correspond thereto. In the first case, passing from Q to an indefinitely near point O, say the red O, we draw from this point to the several points V a set of barriers, say the red barriers; while in the second case, passing from Q to an indefinitely near point O, say the blue O, we draw from this point to the several points V a set of barriers, say the blue barriers; and we then proceed as before, viz., in the first case, drawing from Q to the point S a curve which does not meet any of the red barriers, we determine accordingly the suffixes (say the red suffixes) of the several corresponding points S'; and in the second case, drawing in like manner from Q to S a curve which does not meet any of the blue barriers, we determine accordingly the suffixes (say the blue suffixes) of the same points S'. Now the curve drawn from Q to S so as not to cut any of the red barriers, and which is used for the determination of the red suffixes of the several points S', will in general cut certain of the blue barriers; and, by examining the suffixes of the blue barriers which are thus cut, we determine the blue suffixes of the same points S'; the result of course depending only on the situation of S in one or other of the regions formed by the red barriers and the blue barriers conjointly. In particular, the point S may be so situate that we can from Q to S draw a curve not meeting any red barrier or any blue barrier; and in this case the red suffixes and the blue suffixes are identical.

17. We may imagine the first plane as consisting of n superimposed planes or sheets, say the sheets 1, 2, ... n. Each barrier 12 is considered as a line drawn in the two sheets 1 and 2; and so on in other cases. The point P is considered as a set of superimposed points P_1, P_2 ... P_n moving in the several sheets respectively; under the convention that P_1 moving in the sheet 1, and coming to cross a barrier 12, passes into the sheet 2 and becomes P_2; and the like in other cases. And this being so, we say that to a point P, considered as a point P_α in the sheet α, there corresponds in the second plane one and only one point P'_α; and that P moving continuously in any manner (subject to the change of sheet as just explained), each of the n corresponding points P' will also move continuously, and so that each such point P'_α will return to its original position, upon the corresponding point P_α returning to its original position and sheet. This is, in fact, Riemann's theory, only instead of the points P' we must speak of the values $x'+iy'$ of the irrational function of $x+iy$.

18. Everything is of course symmetrical as regards the two planes; we have therefore, in the second plane, a system of points V' and of barriers, and in the first plane a system of points (W) and of counter-barriers. To a given point P' in the second plane there correspond m points P in the first plane; and we can (the determination depending on the system of barriers in the second plane) assign to the m points suffixes, thereby distinguishing them as the corresponding points P_1, P_2, ... P_m. And we may imagine the second plane as consisting of m superimposed planes or sheets, say the sheets 1, 2, 3 ... m; the general theorem then is that to a point P or P' in either plane, considered as a point P_α or $P'_{\alpha'}$ in the sheet α or α', there corresponds in the other plane one and only one point P'_α or $P_{\alpha'}$; and that the first-mentioned point in either plane moving continuously in any manner (subject to the proper change of sheet), the corresponding point in the other plane will also move continuously, and will return to its original position and sheet, upon the first-mentioned point returning to its original position and sheet.

19. In all that precedes it has been assumed that to a branch-point V there correspond two united points represented by (W'), and $n-2$ distinct points W'; the cases of a point (W') composed of three or more united points, or of the points W' uniting themselves in sets in any other manner, would give rise to further specialities.

January 10th, 1878.

Lord RAYLEIGH, F.R.S., President, in the Chair.

Mr. F. B. W. Phillips, B.A., was elected a member, and Mr. R. R. Webb, M.A., was admitted into the Society.

The following papers were read :—

"On the meaning of the Differential Symbol D^n, when n is fractional:" Mr. J. Hammond. (Professors Cayley and Tanner took part in a discussion on this paper.)

"On Partial Differential Equations with several dependent Variables:" Prof. Lloyd Tanner.

"On the relation between the Functions of Laplace and Bessel:" Lord Rayleigh.

Mr. S. Roberts communicated a result bearing upon his paper read at the December Meeting; and Prof. Cayley gave an expression for the Surface of an Ellipsoid, communicated to him by Prof. Tait. The Chairman, Profs. Cayley and Tanner, and Mr. Webb made a few remarks on the subject.

The following presents were received:—

"Proceedings of Royal Society of Edinburgh," ix. 2.

"Ricerche sulle Equazioni Differenziali a primitiva generale algebrica," par Felice Casorati. (R. Accad. dei Lincei, Roma, 1877.)

"Sulle condizioni alle quali deve soddisfare una primitiva, affinche il grado della corrispondente equazione differenziale rispetto alle variabili, riesca minore del normale," comunicazione del M E. Prof. F. Casorati. (R. Ist. Lombardo, 29 Nov., 1877): from the Author.

"Bulletin des Sciences Mathématiques," Oct. 1877.

"Crelle," Band 84, 2es and 3es Heften; Berlin, 1877.

"Proceedings of Royal Society," No. 184, Vol. xxvi.

"Transactions of the Connecticut Academy," Vol. iv., Pt. 1; Newhaven, 1877.

"S. K. M. Conferences held in connection with the special Loan Collection of Scientific Apparatus, 1876—Physics and Mechanics; Chemistry, Biology, Physical Geography, Geology, Mineralogy, and Meteorology, 1876" (two volumes).

On Partial Differential Equations of the First Order with several dependent variables. By H. W. Lloyd Tanner, M.A.

[*Read January 10th*, 1878.]

1. Let

$$y_1, y_2, \ldots\ldots y_n$$

be n functions of m independent variables

$$x_1, x_2, \ldots\ldots x_m,$$

m being not less than n. The equations proposed for consideration in the present paper are those involving the differential coefficients

$$\frac{dy_i}{dx_j} (i = 1, \ldots n;\ j = 1, \ldots m).$$

It is desirable to show that such equations can always be reduced to a certain standard form, and this will be done in Arts. 2—4. Before attempting the transformation in question, it is desirable to express the given equation as far as possible in terms of Jacobians. For instance, if the expression

$$\frac{dy_1}{dx_1} \cdot \frac{dy_2}{dx_2} - \frac{dy_1}{dx_2} \cdot \frac{dy_2}{dx_1}$$

occurred in the given equation, we should regard it, not as composed of two terms of the second degree, but as a single term

$$\frac{d(y_1, y_2)}{d(x_1, x_2)}$$

of the first degree. Similarly, the determinant

$$\begin{vmatrix} \frac{dy_1}{dx_1}, & \frac{dy_1}{dx_2}, & \ldots\ldots & \frac{dy_1}{dx_r} \\ \frac{dy_2}{dx_1}, & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dy_r}{dx_1}, & \ldots & \ldots & \frac{dy_r}{dx_r} \end{vmatrix}$$

would be regarded as a single term of the first degree, and will be indicated as usual by the symbol

$$\frac{d(y_1 \ldots y_r)}{d(x_1 \ldots x_r)}.$$

It will be assumed that this preliminary reduction has been effected; for, though not essential, it has a very important influence upon the formal simplicity of the transformed equation.

Reduction to the Homogeneous Form.

2. By a homogeneous equation we understand one which has the following properties :—

(1.) All the Jacobians involved are of the n^{th} order; n being the number of dependent variables.

(2.) The equation is algebraically homogeneous with respect to these Jacobians.

(3.) Besides the Jacobians the equation involves only the independent variables.

It may be added that the common degree of the terms of the homogeneous equation is never greater than the degree of any of the non-homogeneous equations to which it is equivalent.

Suppose we are given an equation, or a system of equations, involving n dependent and m independent variables: but it is not assumed that all of these variables are present in each equation. Let $u_1, \dots u_n$ be n independent functions of $x_1 \dots x_m$, $y_1 \dots y_n$; such that

$$u_i = 0 \ (i = 1 \dots n) \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

are particular solutions of the given equation or system of equations. It will be shown that the transformed system in which $u_1 \dots u_n$ are the dependent variables, and $x_1 \dots x_m$, $y_1 \dots y_n$ the independent variables, will be composed of homogeneous equations.

Differentiate any one of the equations (1) with respect to any x; we get

$$\frac{du_i}{dx_j} + \frac{du_i}{dy_1} \cdot \frac{dy_1}{dx_j} + \dots + \frac{du_i}{dy_n} \cdot \frac{dy_n}{dx_j} = 0,$$

or

$$-\frac{du_i}{dx_j} = \frac{du_i}{dy_1} \cdot \frac{dy_1}{dx_j} + \dots + \frac{du_i}{dy_n} \cdot \frac{dy_n}{dx_j} \dots\dots\dots\dots\dots (2).$$

Now we know that, if

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj},$$

then

$$\begin{vmatrix} c_{11}, & c_{12}, & \dots & c_{1n} \\ c_{21}, & c_{22}, & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}, & c_{n2}, & \dots & c_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}, & a_{12}, & \dots & a_{1n} \\ a_{21}, & a_{22}, & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11}, & b_{12}, & \dots & b_{1n} \\ b_{21}, & b_{22}, & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1}, & b_{n2}, & \dots & b_{nn} \end{vmatrix}$$

Comparing this with (2), we have

$$(-)^n \frac{d(u_1 \dots u_n)}{d(\xi_1 \dots \xi_n)} = \frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)} \cdot \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} \dots\dots\dots (3),$$

where $\xi_1 \dots \xi_n$ represent any n of the m quantities $x_1 \dots x_m$.

A similar result is obtained when some of the ξ's are replaced by

some of the y's. For we have identically

$$\frac{du_i}{dy_j} = \frac{du_i}{dy_1}.0 + \ldots + \frac{du_i}{dy_j}.1 + \ldots + \frac{du_i}{dy_n}.0$$
$$= \frac{du_i}{dy_1}.\frac{dy_1}{dy_j} + \ldots + \frac{du_i}{dy_j}.\frac{dy_j}{dy_j} + \ldots + \frac{du_i}{dy_n}.\frac{dy_n}{dy_j} \ldots\ldots\ldots\ldots (4),$$

if we regard the y's as mutually independent, so that $\frac{dy_i}{dy_j}$ vanishes unless $i=j$, when it is unity.

Now take r equations (2), the r values of x_j being called $\xi_1 \ldots \xi_r$; and $n-r$ equations (4), calling the selected y's, $\eta_{r+1}, \ldots \eta_n$. In these n equations let u_i be replaced by $u_1 \ldots u_n$, in succession. Then we have n^2 equations which give

$$(-)^r \frac{d(u_1 \ldots u_r, u_{r+1} \ldots u_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)} = \frac{d(u_1 \ldots u_n)}{d(\eta_1 \ldots \eta_n)} . \frac{d(\eta_1 \ldots \eta_r, \eta_{r+1} \ldots \eta_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)}.$$

Here, $\eta_1 \ldots \eta_n$ and $y_1 \ldots y_n$ are the same quantities arranged in a different order. Let this arrangement be such that one order may be reduced to the other by an even number of transpositions. If this be not the case, it is only necessary to transpose a single pair of $\eta_1 \ldots \eta_r$, *or* of $\eta_{r+1} \ldots \eta_n$ to bring it about. Then we have

$$\frac{d(u_1 \ldots u_n)}{d(\eta_1 \ldots \eta_n)} = \frac{d(u_1 \ldots u_n)}{d(y_1 \ldots y_n)}.$$

Also, in virtue of the hypothesis as to the values of $\frac{dy_i}{dy_j}$, we have

$$\frac{d(\eta_1 \ldots \eta_r, \eta_{r+1} \ldots \eta_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)} = \frac{d(\eta_1 \ldots \eta_r)}{d(\xi_1 \ldots \xi_r)},$$

as may easily be seen by writing it in the form of a determinant. Hence

$$(-)^r . \frac{d(u_1 \ldots u_r, u_{r+1} \ldots u_n)}{d(\xi_1 \ldots \xi_r, \eta_{r+1} \ldots \eta_n)} = \frac{d(u_1 \ldots u_n)}{d(y_1 \ldots y_n)} . \frac{d(\eta_1 \ldots \eta_r)}{d(\xi_1 \ldots \xi_r)} \ldots\ldots (5),$$

an equation which includes (3) as a particular case.

In the second factor on the right side of (5), $\eta_1 \ldots \eta_r$ may be any r of the n variables $y_1 \ldots y_n$; and $\xi_1 \ldots \xi_r$ may be any r of the m variables $x_1 \ldots x_m$. Thus this factor is any Jacobian of the r^{th} order: that is, since r may have any value from 1 to n inclusive, it is any Jacobian of any order that can be formed from $y_1 \ldots y_n$ with respect to $x_1 \ldots x_m$. By (5), then, any such Jacobian is expressed as a ratio between two Jacobians of the n^{th} order; the denominator always being

$$\frac{d(u_1 \ldots u_n)}{d(y_1 \ldots y_n)}, = \Delta, \text{ say.}$$

If, then, we substitute in any of the original equations from (5), we get an equation in which all the Jacobians are of the n^{th} order.

Again, each term of the transformed equation is of degree zero with respect to these Jacobians: but if in the original equation any term is of the r^{th} degree, the denominator of the corresponding term in the transformed equation is Δ^r. Suppose the degree of the original equation to be r. Then we can clear transformed equation of fractions by multiplying by Δ^r. But by this the transformed equation is made algebraically homogeneous, of the r^{th} degree.

Lastly, the transformed equation, besides the Jacobians of u, will involve only the variables $x_1 \ldots x_m$, $y_1 \ldots y_n$, which are the independent variables of the transformed equation.

But these characters are just those by which we defined the homogeneous form of equation—so that the required reduction has been effected.

3. In some cases—more especially when the equations are linear—another method is applicable, which involves a change in the dependent variables only, not increasing the number of independent variables. Say, the original equation has the n dependent variables $y_1 \quad y_n$ and the m independent variables $x_1 \ldots x_m$. Suppose also that, when $\eta_1 \ldots \eta_n$ are taken as dependent variables, the equation becomes homogeneous; $\eta_1 \ldots \eta_n$, being functions of $x_1 \ldots x_m$, $y_1 \ldots y_n$, such that

$$\frac{d\,(\eta_1 \ldots \eta_n)}{d\,(y_1 \ldots y_n)}$$

does not vanish. Then we may write

$$\frac{d\eta_i}{dx_j} = \frac{d\eta_i}{dy_1} \cdot \frac{dy_1}{dx_j} + \ldots\ldots + \frac{d\eta_i}{dy_n} \cdot \frac{dy_n}{dx_j} + \frac{d\eta_i}{dx_1} \cdot \frac{dx_1}{dx_j} + \ldots\ldots + \frac{d\eta_i}{dx_m} \cdot \frac{dx_m}{dx_j},$$

since $\dfrac{dx_i}{dx_j}$ vanishes unless $i = j$, when it becomes unity.

From these equations we infer, by a generalization of the rule for multiplying determinants, that

$$\frac{d\,(\eta_1 \ldots \eta_n)}{d\,(\xi_1 \ldots \xi_n)} = \Sigma \frac{d\,(\eta_1 \ldots \eta_n)}{d\,(v_1 \ldots v_n)} \cdot \frac{d\,(v_1 \ldots v_n)}{d\,(\xi_1 \ldots \xi_n)} \quad \ldots\ldots\ldots\ldots\ldots(6),$$

where $\xi_1 \ldots \xi_n$ are any n of $x_1 \ldots x_m$; $v_1 \ldots v_n$ are any n of $y_1 \ldots y_n$, $\xi_1 \ldots \xi_n$;* and the sign of summation includes all the different sets of the v's. The second factor on the right-hand side reduces to a

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* If any v were an x not included in $\xi_1 \ldots \xi_n$, the second factor on the right-hand side would vanish identically.

Jacobian of the r^{th} order when $v_1 \dots v_n$ include r of $y_1 \dots y_n$ and $n-r$ of $\xi_1 \dots \xi_n$.

Suppose that the y-equation is linear: say it is

$$P \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} + P' \frac{d(y_1 \dots y_n)}{d(\xi_1' \dots \xi_n')} + \dots\dots = 0 \quad \dots\dots\dots (y),$$

and let the η-equation be

$$Q \frac{d(\eta_1 \dots \eta_n)}{d(\xi_1 \dots \xi_n)} + Q' \frac{d(\eta_1 \dots \eta_n)}{d(\xi_1' \dots \xi_n')} + \dots\dots = 0 \quad \dots\dots\dots (\eta),$$

the second equation differing from the first in not involving any Jacobians of an order lower than the n^{th}. The left-hand side of (η) is, by (6),

$$= \frac{d(\eta_1 \dots \eta_n)}{d(y_1 \dots y_n)} \left\{ Q \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} + Q' \frac{d(y_1 \dots y_n)}{d(\xi_1' \dots \xi_n')} + \dots\dots \right\}$$

$$+ \text{terms involving } y\text{-Jacobians of lower orders.}$$

Comparing with (y), we get

$$\frac{Q}{P} = \frac{Q'}{P'} = \dots\dots$$

Hence, as we are not concerned with the absolute values of $Q, Q', \dots$, but only with their ratios, we may say that in the η-equation the coefficients $Q, Q', \dots$ are equal to the coefficients of the corresponding y-Jacobians in the given equation (y). By comparing the coefficients of the lower Jacobians in (y) with those in (η) after the application of (6), we get a system of equations for $\eta_1 \dots \eta_n$. If we can get n particular solutions of this system, the transformation is effected.

4. As an example, consider the equation

$$\frac{d(y_1, y_2)}{d(x_1, x_2)} + \frac{d(y_1, y_2)}{d(x_2, x_3)} + \frac{y_2}{x_3} \cdot \frac{dy_1}{dx_2} - \frac{y_2}{x_1} \cdot \frac{dy_2}{dx_1}$$

$$+ \left(\frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3} \right) \frac{dy_2}{dx_2} + \frac{y_2}{x_1} \cdot \frac{dy_2}{dx_3} + \frac{y_2^2}{x_1 x_3} = 0.$$

We have to find η_1, η_2 so that, if possible, the above may reduce to

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_3)} = 0.$$

A comparison of coefficients gives the following equations for η_1, η_2:

$$\frac{d(\eta_1, \eta_2)}{d(y_1, x_2)} = 0, \qquad \frac{d(\eta_1, \eta_2)}{d(y_2, x_2)} + \frac{y_2}{x_1} \cdot \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_1)} + \frac{d(\eta_1, \eta_2)}{d(y_1, x_3)} - \frac{y_2}{x_3} \cdot \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_2)} + \frac{d(\eta_1, \eta_2)}{d(y_2, x_3)} - \left(\frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3}\right) \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_3)} - \frac{y_2^2}{x_1 x_3} \cdot \frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0.$$

These equations will be solved in the sequel. For the present it suffices to give a particular solution,

$$\eta_1 = x_1 y_1 + x_2 y_2, \quad \eta_2 = x_3 y_2,$$

which is verified without difficulty. The given equation is therefore equivalent to

$$\frac{d(x_1 y_1 + x_2 y_2, x_3 y_2)}{d(x_1, x_2)} + \frac{d(x_1 y_1 + x_2 y_2, x_3 y_2)}{d(x_2, x_3)} = 0.$$

The above example will illustrate an important point, namely, that even when the complete reduction sought is not attainable, yet the number of terms may be considerably reduced, and the equations proportionally simplified. For suppose the second member of the original equation to be Y instead of zero; then the η-equation would be of the form

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_3)} = H.$$

Thus the seven terms of the original equation would be reduced to three.

Another point of importance arises. If the y-equation of the last example be transformed into a homogeneous equation by the general method, the new equation will consist of seven terms. Suppose now that the change of variables above indicated be made in the new equation. This will reduce to two terms, thus showing that in a homogeneous equation a change of the independent variables may cause a considerable simplification. To this point I hope to return hereafter.

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5. The results of the preceding Articles enable us to confine our attention to homogeneous equations. We shall further restrict ourselves to linear equations—viz., equations which are included in the form

$$\Sigma P \cdot \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots (7),$$

where $x_1 \dots x_n$ are any n of the m independent variables; P is any function of $x_1 \dots x_m$; and the sign of summation extends to the $\frac{\lfloor m}{\lfloor n \, \lfloor m-n}$ terms which can be formed by taking different sets of $x_1 \dots x_n$.

Of this equation (7) we shall discuss three cases:

1) When it consists of a single term, or is reducible to this form.
2) When it consists of two terms, or is reducible to this form.
3) When only certain specified terms occur.

Under each case will be discussed the systems of simultaneous equations corresponding thereto.

Single Term Equations.

6. The simplest form of the general equation (7), in which, namely, only one term occurs, is

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8).$$

The general solution is $\phi(y_1 \dots y_n) = 0$ ..... (9),

where ϕ is an arbitrary function, involving, besides its expressed arguments, any quantity not explicitly involved in (8). If, for instance, $y_1 \dots y_n$ are functions of $x_1 \dots x_n$, $x_{n+1} \dots x_m$, the solution of (8) may be written $\phi(y_1 \dots y_n, x_{n+1} \dots x_m)$ ..... (10).

Let us now seek the conditions under which the general equation (7) is reducible to this well-known form—viz., when it may be written

$$\frac{d(y_1 \dots y_n, u_1 \dots u_p)}{d(x_1 \dots \dots \dots x_m)} = 0 \quad (m = n+p) \dots\dots\dots\dots (11).$$

This implies $\frac{\lfloor n+p}{\lfloor n \lfloor p}$ equations of the form

$$P = \pm\lambda \cdot \frac{d(u_1 \dots \dots u_p)}{d(x_{n+1} \dots x_{n+p})} \quad \dots\dots\dots\dots\dots\dots (12),$$

by means of which the $\frac{\lfloor n+p}{\lfloor n \lfloor p}$ coefficients P are expressed in terms of $p+1$ independent quantities $\lambda, u_1 \dots u_p$. The former number is greater than the latter, so that the P's are subject to certain conditions when the transformation is possible, save when either n or p is zero or when $n=1$. In the first case there is no dependent variable; in the second case the equation is (8) itself; in the third case we have only one dependent variable, or the equation is a common partial differential equation of the first order. Except in these cases the P's must satisfy certain relations which it is our object to discover. Supposing these relations to be satisfied, we then require the forms of $u_1 \dots u_p$. To attain the former we might eliminate $\lambda, u_1 \dots u_p$ from the equations (12), and find $u_1 \dots u_2$ by solving the system formed by eliminating λ; but we prefer to avail ourselves of the peculiar form of (12), and to make use of an artifice by means of which both ends may be attained without difficulty.

It is convenient to assume that $\xi_1 \dots \xi_m$ are the same as $x_1 \dots x_m$, but arranged in a different order. Let $\xi_1 \dots \xi_m$ be so chosen that in

(7) the coefficient of $\dfrac{d(y_1 \dots y_n)}{d(\xi_{1+p} \dots \xi_{n+p})}$ ..... (13)

is not zero, and let the equation be divided throughout by this coefficient.

Consider next the p Jacobians formed from (13) by replacing ξ_{p+1} by $\xi_1, \xi_2 \dots \xi_p$ respectively. Let the coefficients of these Jacobians in (7) be denoted by $P^1_{p+1}, P^2_{p+1}, \dots P^p_{p+1}$ respectively, the suffix denoting the ξ of (13) which is removed, the index denoting the ξ which replaces it. Then a comparison of (7) and (11) gives

$$1 = \lambda \frac{d(u_1 \dots u_p)}{d(\xi_1 \dots \xi_p)},$$

$$P^p_{p+1} = -\lambda \frac{d(u_1 \dots u_p)}{d(\xi_1 \dots \xi_{p-1}, \xi_{p+1})},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$P^1_{p+1} = \pm\lambda \frac{d(u_1 \dots u_p)}{d(\xi_2 \dots \xi_{p+1})},$$

equations which may be conveniently expressed in the form

$$\pm P^1_{p+1}, \mp P^2_{p+1}, \dots -P^p_{p+1}, 1 = \lambda \frac{d(u_1 \dots u_p)}{d(\xi_1 \dots \xi_{p+1})} \quad \dots\dots (14),$$

which is meant to imply that the i^{th} member on the left is equal to the determinant formed on the right by erasing the i^{th} constituent of the denominator.

Now the equation $\dfrac{d(u, u_1, \dots u_p)}{d(\xi_1, \dots \xi_{p+1})} = 0$

is identically satisfied by $u = u_1, u_2 \dots u_n$. This equation may be written

$$\frac{du}{d\xi_1} \cdot \frac{d(u_1 \dots u_p)}{d(\xi_2 \dots \xi_{p+1})} - \frac{du}{d\xi_2} \cdot \frac{d(u_1 \dots u_p)}{d(\xi_1, \xi_3 \dots \xi_{p+1})} + \dots\dots$$

$$\pm \frac{du}{d\xi_{p+1}} \cdot \frac{d(u_1 \dots u_p)}{d(\xi_1 \dots \xi_p)} = 0,$$

or, because of (14),

$$P^1_{p+1} \frac{du}{d\xi_1} + P^2_{p+1} \frac{du}{d\xi_2} + \dots\dots + P^p_{p+1} \frac{du}{d\xi_p} + \frac{du}{d\xi_{p+1}} = 0.$$

This, then, is an equation which must be satisfied by $u_1 \dots u_p$; but evidently it is insufficient of itself to determine them, since it gives no information as to how they depend upon $\xi_{p+2} \dots \xi_m$. To supply this deficiency we form other equations, starting in each case with the

Jacobian (13), but replacing, for one equation ξ_{p+2}, for another ξ_{p+3}, and so on. Thus we form a system of n equations, viz.:

$$\left.\begin{aligned}
\Delta_{p+1}u &= P^1_{p+1}\frac{du}{d\xi_1}+\ldots\ldots+P^p_{p+1}\frac{du}{d\xi_p}+\frac{du}{d\xi_{p+1}}=0\\
\Delta_{p+2}u &= P^1_{p+2}\frac{du}{d\xi_1}+\ldots\ldots+P^p_{p+2}\frac{du}{d\xi_p}+\frac{du}{d\xi_{p+2}}=0\\
\ldots\quad &\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
\Delta_{p+n}u &= P^1_{p+n}\frac{du}{d\xi_1}+\ldots\ldots+P^p_{p+n}\frac{du}{d\xi_p}+\frac{du}{d\xi_{p+n}}=0
\end{aligned}\right\}\quad\ldots\ldots\ldots(15),$$

which must be satisfied by $u_1 \ldots u_p$.

But the system (15) comprises n equations, which are mutually independent, since each contains a differential coefficient not occurring in any of the others: there are $n+p$ independent variables. Hence (15) cannot have more than p common solutions, nor can it have so many, unless, identically,

$$\Delta_r P^i_s = \Delta_s P^i_r \quad (i=1\ldots p;\ r,s=p+1,\ldots p+n)\ldots\ldots\ldots(16),$$

the Δ's being as defined in (15).

When these conditions are fulfilled, the common integrals of (15) are given by integrating

$$\left.\begin{aligned}
d\xi_1-P^1_{p+1}d\xi_{p+1}-P^1_{p+2}d\xi_{p+2}-\ldots\ldots-P^1_{p+n}d\xi_{p+n}&=0\\
d\xi_2-P^2_{p+1}d\xi_{p+1}-\ldots\ldots\ldots\ldots\ldots-P^2_{p+n}d\xi_{p+n}&=0\\
\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots&\\
d\xi_p-P^p_{p+1}d\xi_{p+1}-\ldots\ldots\ldots\ldots\ldots-P^p_{p+n}d\xi_{p+n}&=0
\end{aligned}\right\}\quad\ldots\ldots(17),$$

which then are reducible to p exact differentials. It does not follow, however, that the p integrals of (17) may be put for $u_1 \ldots u_p$. For these integrals are determined from a knowledge of $np+1$ of the P's, and without any reference to the remaining $\frac{\lfloor n+p}{\lfloor n \lfloor p} - np - 1$; and if $u_1 \ldots u_p$ are the p integrals of (17), then the remaining P's are determined without ambiguity from the P's used in (15) or (17). These other P's must therefore satisfy certain conditions which may be determined either by forming other equations similar to (15), and expressing that all such equations must be consequences of (15), or by making use of the known relations between the determinants of the matrix $\dfrac{d(u_1 \ldots u_p)}{d(x_1 \ldots x_{n+p})}$.

But practically it is most convenient in any given case to remember that the integrals of (17) include all the values of u which can be used in (11); and by comparison with the given equation (7) to determine

how many, and which, of these integrals are suitable for our purpose. If there be p integrals of (17), and all of these be suitable, the general solution of (7) will be

$$\phi\,(y_1 \dots y_n, u_1 \dots u_p) = 0 \quad \dots\dots\dots\dots\dots\dots \quad (18),$$

$u_1 \dots u_p$ being in this case the p integrals of (17). If, on the other hand, there are less than p integrals of (17), or all the integrals are not available, then we get a solution,

$$\phi\,(y_1 \dots y_n \cdot u_1 \dots u_{p'}) = 0 \dots\dots\dots\dots\dots\dots\dots (19),$$

where $p' < p$. This is not a general solution, for, as will appear in the sequel, it will satisfy not only (7), but $p-p'$ other differential equations.

The knowledge of this solution is however useful, as it enables us to reduce the number of independent variables by p'. This is effected by taking a new set of independent variables, say $\xi_1 \dots \xi_m$, of which p' are to be $u_1 \dots u_{p'}$, say $\xi_1 = u_1, \dots \xi_{p'} = u_{p'}$; for clearly

$$\frac{d\,(u_1 \dots u_{p'},\ y_1 \dots y_n)}{d\,(u_1 \dots u_{p'},\ \xi_{p'+1} \dots \xi_{p'+n})} = \frac{d\,(y_1 \dots y_n)}{d\,(\xi_{p'+1} \dots \xi_{p'+n})},$$

and so on; so that of the new independent variables only $\xi_{p'+1} \dots \xi_{p'+n}$ will remain.

7. A few simple examples may be useful.

$$\frac{d\,(y_1, y_2)}{d\,(x_1, x_3)} - \frac{d\,(y_1, y_2)}{d\,(x_1, x_4)} - \frac{d\,(y_1, y_2)}{d\,(x_2, x_3)} + \frac{d\,(y_1, y_2)}{d\,(x_2, x_4)} = 0.$$

Let us put $\xi_1 = x_1, \quad \xi_2 = x_3, \quad \xi_3 = x_2, \quad \xi_4 = x_4.$

Then $P_3^2 = 0, \qquad P_3^1 = -1,$

$P_4^2 = -1, \qquad P_4^1 = 0.$

Thus (17) become

$$d\xi_1 + d\xi_3 = 0, \quad \xi_1 + \xi_3 \text{ or } x_1 + x_2 = u_1;$$
$$d\xi_2 + d\xi_4 = 0, \quad \xi_2 + \xi_4 \text{ or } x_3 + x_4 = u_2.$$

Hence the required solution is

$$\phi\,(x_1 + x_2,\ x_3 + x_4,\ y_1,\ y_2) = 0.$$

[It seems worth while to show *à posteriori* how such an integral equation satisfies the proposed differential equation; viz., the integral equation is $y_2 = \phi\,(y_1, x_1 + x_2, x_3 + x_4)$,

so that $dy_2 = A dy_1 + B\,(dx_1 + dx_2) + C\,(dx_3 + dx_4)$;

i. e.,

$$\frac{dy_2}{dx_1} = A\frac{dy_1}{dx_1} + B, \qquad \frac{dy_2}{dx_3} = A\frac{dy_1}{dx_3} + C,$$
$$\frac{dy_2}{dx_2} = A\frac{dy_1}{dx_2} + B, \qquad \frac{dy_2}{dx_4} = A\frac{dy_1}{dx_4} + C;$$

or, eliminating A, B, C,

$$\frac{\frac{dy_2}{dx_1} - \frac{dy_2}{dx_2}}{\frac{dy_1}{dx_1} - \frac{dy_1}{dx_2}} = \frac{\frac{dy_2}{dx_3} - \frac{dy_2}{dx_4}}{\frac{dy_1}{dx_3} - \frac{dy_1}{dx_4}}.]$$

The equation of Art. 4, viz.,

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_3)} = 0$$

is found in the same way to have for its solution

$$\phi(x_1 + x_3,\ \eta_1,\ \eta_2) = 0,$$

so that the general solution of the y-equation of that Art. is

$$\phi(x_1 + x_3,\ x_1 y_1 + x_2 y_2,\ x_3 y_2) = 0.$$

As a third example, take

$$\frac{d(y_1, y_2, y_3)}{d(x_1, x_2, x_3)} + \frac{d(y_1, y_2, y_3)}{d(x_3, x_4, x_5)} + \frac{d(y_1, y_2, y_3)}{d(x_2, x_4, x_6)} = 0.$$

Here let $\xi_4 = x_2, \quad \xi_5 = x_4, \quad \xi_6 = x_6,$

$\xi_1 = x_1, \quad \xi_2 = x_3, \quad \xi_3 = x_5.$

On trying, we find that every P_j^i vanishes; so that (17) would become

$$d\xi_1 = 0, \quad d\xi_2 = 0, \quad d\xi_3 = 0;$$

or $$u_1 = x_1, \quad u_2 = x_3, \quad u_3 = x_5.$$

But without trial we know at once that these cannot be available; for, if they were, x_1, x_3, x_5 would not occur amongst the independent variables of the proposed equation. [*Cf.* (10).]

This, then, is an equation not soluble in the form (18) or (19). By another method we shall get its solution hereafter (Art. 14).

8. In the preceding paragraphs it has been assumed that the P's do not involve the dependent variables. It may happen, however, that we have to deal with an equation which is 'homogeneous' except in this respect, and the trouble of reducing to the homogeneous form may be avoided in many cases. Suppose, then, that some or all of the P's involve $y_1 \dots y_n$, so that the same will be true of $u_1 \dots u_p$. Now, in forming the determinant

$$\frac{d(y_1 \dots y_n, u_1 \dots u_p)}{d(x_1 \dots x_{n+p})},$$

we differentiate $u_1 \dots u_p$ as if the y's were constant. For let any constituent of the determinant be

$$\frac{du_i}{dx_j} + \frac{du_i}{dy_1} \cdot \frac{dy_1}{dx_j} + \dots + \frac{du_i}{dy_n} \cdot \frac{dy_n}{dx} \quad \dots\dots\dots\dots\dots\dots (19).$$

In the same column we shall have the constituents

$$\frac{dy_1}{dx_j}, \quad \frac{dy_2}{dx_j}, \quad \ldots\ldots \quad \frac{dy_n}{dx_j}.$$

Multiply the rows in which these constituents occur by

$$-\frac{du_i}{dy_1}, \quad -\frac{du_i}{dy_2}, \quad \ldots\ldots, \quad -\frac{du_i}{dy_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

respectively, and add to the row to which (19) belongs. This, of course, will not affect the value of the determinant, and it will reduce (19) to

$$\frac{du_i}{dx_j}.$$

But observing that the multipliers (20) are the same for all values of j, we find that all the constituents in the row of (19) are similarly reduced. Applying the same in turn to all the rows involving u, we get the result announced—viz., that in forming

$$\frac{d\,(y_1 \ldots y_n, u_1 \ldots u_p)}{d\,(x_1 \ldots\ldots\ldots x_{n+p})},$$

we may regard $y_1 \ldots y_n$ as constants when occurring in $u_1 \ldots u_p$.

Hence, in seeking the integrals of (15) or (17), we may regard any y which occurs in the coefficients as a constant.

Simultaneous Mononomial Equations.

9. These equations have for their general solution

$$\phi\,(y_1 \ldots y_n, u_1 \ldots u_p) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (20);$$

but $n+p$ is now less than the number of independent variables. The system of differential equations which has (20) for its general solution may be written

$$\frac{d\,(y_1 \ldots y_n, u_1 \ldots u_p)}{d\,(x_1 \ldots\ldots\ldots x_m)} = 0 \quad (m > n+p) \quad \ldots\ldots\ldots\ldots (21).$$

Let q be the number of independent equations that can be formed from (21). Then we have

$$\left.\begin{aligned} q &= m-(n+p)+1 \\ \text{or} \qquad m+1 &= n+p+q \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

For if we take the equations formed by retaining

$$\begin{aligned}
&x_1 \ldots\ldots x_{n+p-1}, x_{n+p}, \\
&x_1 \ldots\ldots x_{n+p-1}, x_{n+p+1}, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&x_1 \ldots\ldots x_{n+p-1}, x_{m-1}, \\
&x_1 \ldots\ldots x_{n+p-1}, x_m,
\end{aligned}$$

supposing $x_1 \dots x_{n+p}$ so chosen that in the first equation the coefficient of

$$\frac{d(y_1 \dots y_n)}{d(x_{p+1} \dots x_{p+n})},$$

that is, the determinant $\dfrac{d(u_1 \dots u_p)}{d(x_1 \dots x_p)}$

does not vanish. Then in the $\overline{i+1}^{\text{th}}$ equation we shall have a Jacobian

$$\frac{d(y_1 \dots y_{n-1}, y_n)}{d(x_{p+1} \dots x_{p+n-1}, x_{p+n+i})},$$

which does not occur in any of the others. Therefore the $m-(n+p)+1$ equations thus formed are independent.

Also in a matrix

$$\left\| \begin{matrix} a_{11}, & a_{12} & \dots\dots & a_{1,\,m+n} \\ \dots & \dots & \dots & \dots \quad \dots \\ a_{m+1,1}, & \dots\dots\dots & & a_{m+1,\,m+n} \end{matrix} \right\|$$

we know that, if the n determinants, formed by combining the first m columns with each of the last n columns in turn, all vanish, then all the determinants vanish, provided

$$\left| \begin{matrix} a_{21} & \dots\dots & a_{2,\,m} \\ \dots & \dots & \dots \quad \dots \\ a_{m+1,1} & \dots\dots & a_{m+1,\,m} \end{matrix} \right|$$

is not zero. Applying this to (21), we learn that, if the $m-(n+p)+1$ equations formed as above all vanish, then every equation of (21) is satisfied—viz., there are *only* $m-(n+p)+1$ independent equations in the system. So that (22) is proved.

10. Suppose now we have a system of q equations each of the form (7). We might take any of the equations involving all the independent variables, and find the most general solution it admits of the form (20); then, by substitution in the other equations, determine the limitations imposed upon ϕ, $u_1 \dots u_p$. But it is more convenient generally to adopt another course. We reduce the given system as nearly as possible to the form of (21) when expanded as explained above. Each equation will contain a Jacobian not occurring in any of the other equations. In these Jacobians we shall have $n-1$ independent variables (say $x_{p+1} \dots x_{p+n-1}$) common to all; and q (say $x_{p+n} \dots x_m$) each of which occurs only in one. The other Jacobians must not involve these q independent variables, but may contain any of the $m-q$ others. This arrangement is always possible if there be a general solution of the form (20).

Having made this arrangement, let the q independent variables, each of which is characterised by occurring in one and only one equation of

the system, be indicated by x_{m-q+1}, x_{m-q+2}, ... x_m; or, what is the same thing, by x_{n+p}, x_{n+p+1}, ... $x_{n+p+q-1}$. To distinguish the equations of the given system conveniently, let that which contains x_{m-q+i} or $x_{n+p+i-1}$ be called the i^{th} equation of the system.

Suppose now that the first equation of the system is reducible to

$$\frac{d\,(y_1 \ldots y_n, u_1 \ldots u_p)}{d\,(x_1 \ldots \ldots x_{n+p})} = 0,$$

and let the x's be so arranged that the coefficient of

$$\frac{d\,(y_1 \ldots y_n)}{d\,(x_{1+p} \ldots x_{n+p})}$$

is not zero. Divide the equation throughout by this coefficient, and indicate the new coefficient of the Jacobian formed by writing x_j for x_{i+p} in the above by ${}_1P^j_{i+p}$,

where the "1" to the left indicates the number of the equation.

We have then

$$\pm {}_1P^1_{i+p},\ \mp {}_1P^2_{i+p},\ \ldots\ldots,\ -{}_1P^p_{i+p},\ 1 = \lambda_1 \frac{d\,(u_1 \ldots u_n)}{d\,(x_1 \ldots x_p, x_{i+p})}.$$

But $u_1 \ldots u_p$ satisfy the equation

$$\frac{d\,(u, u_1 \ldots u_p)}{d\,(x_1 \ldots x_p, x_{i+p})} = 0$$

identically. That is, they satisfy the equations

$${}_1P^1_{i+p}\frac{du}{dx_1} + {}_1P^2_{i+p}\frac{du}{dx_2} + \ldots\ldots + {}_1P^p_{i+p}\frac{du}{dx_p} + \frac{du}{dx_{i+p}} = 0 \ldots\ldots (15')$$
$$(i = 1, 2, \ldots n).$$

The n equations (15′) are not sufficient to determine u, for they do not say how u depends upon $x_{n+p+1} \ldots x_m$. To supply this knowledge we form from each of the remaining $q-1$ equations of the given system one other equation for u. From the i^{th} equation, for instance, we get

$${}_iP^1_{m-q+i}\frac{du}{dx_1} + \ldots\ldots + {}_iP^p_{m-q+i}\frac{du}{dx_p} + \frac{du}{dx_{m-q+i}} = 0 \ldots\ldots\ldots (15'')$$
$$(i = 2, 3, \ldots q).$$

It is here tacitly assumed that the coefficient of

$$\frac{d\,(y_1 \ldots y_n)}{d\,(x_{1+p} \ldots x_{n-1+p}, x_{n-1+p+i})}$$

does not vanish; but the coefficient in question being

$$\lambda_i \frac{d\,(u_1 \ldots u_p)}{d\,(x_1 \ldots x_p)},$$

it appears that this is no new assumption.

If in (15′) i take all values from 1 to n, and in (15″) all values from 2 to q [the value n in (15′) and 1 in (15″) would give the same equation], we get a system of $n+q-1$ independent equations involving m independent variables. These can at most have $m-(n+q)+1$, or p solutions. Thus, if $u_1 \dots u_p$ exist, they are determined by (15′), (15″), or by integrating the p expressions

$$dx_i - {}_1P^i_{1+p}\,dx_{1+p} - \dots - {}_1P^i_{n+p}\,dx_{n+p} - {}_2P^i_{n+p+1}\,dx_{n+p+1} - \dots - {}_qP^i_m\,dx_m = 0 \quad (i = 1, 2, \dots p). \qquad \dots\dots(17').$$

In order that (17′) should have p integrals, certain conditions involving the differential coefficients of the P's must be satisfied, as (16). In order that the P's not included in the above system (17′), or (15′), (15″), may be consistent, they must satisfy certain algebraical conditions. Of these latter, one is obviously that ${}_jP^i_{k+p}$ is the same for all values of j, k having one of the values 1, 2, ... $n-1$, $n-1+j$.

If any of these conditions be not satisfied, then there are less than p values of u; and there is, consequently, no general solution of the given system of the form (20). But the knowledge of p' $(p' < p)$ values of u enables us to simplify the system, reducing the number of independent variables by p'.

Binomial Equations.

11. The most general form of this equation is

$$\frac{d(u_1 \dots u_m)}{d(x_1 \dots x_m)} + P\frac{d(u_1 \dots u_n,\ u_{n+1} \dots u_m)}{d(y_1 \dots y_n,\ x_{n+1} \dots x_m)},$$

which, by (5), reduces to

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = P \qquad \dots\dots(24),$$

P being a function of x, y. I have not been able to obtain a general solution of this equation; but it is easy to get, in two distinct ways, as many solutions as we please.

12. In (24) let us replace $x_1 \dots x_n$ by a new set of independent variables $\xi_1 \dots \xi_n$. Then we have

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} \cdot \frac{d(\xi_1 \dots \xi_n)}{d(x_1 \dots x_n)} = P.$$

Say now we attribute to $y_1 \dots y_{n-1}$ any values whatever in terms of $x_1 \dots x_n$, and replace $\xi_1 \dots \xi_{n-1}$ by these values. Then the last-written equation becomes

$$\frac{d(\xi_1 \dots \xi_n)}{d(x_1 \dots x_n)} \cdot \frac{dy_n}{d\xi_n} = P \qquad \dots\dots(25).$$

Lastly, take any convenient value for ξ_n such that the left-hand member does not vanish. By means of these suppositions

$$\frac{d(\xi_1 \dots \xi_n)}{d(x_1 \dots x_n)}$$

becomes a known function of $x_1 \dots x_n$; while P is a given function of $y_1 \dots y_n$, $x_1 \dots x_n$. Both these quantities may therefore be expressed in terms of y_n, $\xi_1 \dots \xi_n$; so that (25) is an ordinary differential equation which gives the value of y_n under the supposed conditions.

As an example, take $$\frac{d(y_1, y_2)}{d(x_1, x_2)} = 1 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (26).$$

Let us find the form of y_2 when $y_1 = x_1^2 + x_2^2$.

Transform to new variables

$$\xi_1 = x_1^2 + x_2^2, \quad \xi_2 = \frac{x_2}{x_1}.$$

Then (24) becomes $$2\left(1 + \frac{x_2^2}{x_1^2}\right)\frac{dy_2}{d\xi_2} = 1;$$

$$\therefore \quad 2\frac{dy_2}{d\xi_2} = \frac{1}{1+\xi_2^2},$$

$$\begin{aligned} 2y_2 &= \phi(\xi_1) + \tan^{-1}\xi_2 \\ &= \phi(x_1^2 + x_2^2) + \tan^{-1}\frac{x_2}{x_1}. \end{aligned}$$

Or, again, take $$\frac{d(y_1, y_2)}{d(x_1, x_2)} = x_1 y_2 - x_2 y_1.$$

Suppose now that $y_1 = \frac{x_1}{x_2}$. Take for new variables

$$\xi_1 = \frac{x_1}{x_2}, \quad \xi_2 = x_1 x_2.$$

Then (24) becomes $$2\frac{x_1}{x_2} \cdot \frac{dy_2}{d\xi_2} = x_1(y_2 - 1);$$

$$2\frac{dy_2}{d\xi_2} = x_2(y_2 - 1) = \sqrt{\frac{\xi_2}{\xi_1}}(y_2 - 1);$$

$$\therefore \quad \frac{2dy_2}{y_2 - 1} = \frac{\xi_2^{\frac{1}{2}}}{\xi_1^{\frac{1}{2}}}\, d\xi_2,$$

whence $$y_2 = 1 + \phi\left(\frac{x_1}{x_2}\right) \cdot e^{\frac{x_1 x_2^2}{3}}.$$

13. It is evident that by attributing thus various forms to $y_1 \dots y_{n-1}$ we can get as many solutions as we please, and that these solutions include all possible solutions of (24) in their number. But the process, though convenient in some cases, is not of much use when $y_1 \dots y_n$ are

restricted to certain forms, and we do not know what the forms may be. In such cases we may get solutions of (24); but $y_1 \dots y_{n-1}$ may have been chosen so as to violate some of the other conditions, and the solutions are useless. It becomes desirable, therefore, to employ a method which does not imply a knowledge of the form of $y_1 \dots y_{n-1}$. This will be now indicated by means of the same particular example,

$$\frac{d(y_1, y_2)}{d(x_1, x_2)} = 1 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (26).$$

Now we have

$$\frac{d(y_1, y_2)}{d(x_1, x_2)} = \frac{d(y_1, y_2)}{d(y_1', y_2')} \cdot \frac{d(y_1', y_2')}{d(y_1'', y_2'')} \dots\dots \frac{d(y_1^{(n)}, y_2^{(n)})}{d(x_1, x_2)} = 1;$$

so that (26) will be satisfied if each of the determinants in the middle term of the last equation is unity. And, moreover, $y_1^{(r)}$, $y_2^{(r)}$ will be particular values of y_1, y_2. Now assume

$$\frac{dy_1^{(n)}}{dx_1} = 0, \quad \frac{dy_1^{(n)}}{dx_2} \cdot \frac{dy_2^{(n)}}{dx_1} = -1 \quad \dots\dots\dots\dots\dots\dots (27),$$

so that

$$\frac{d(y_1^{(n)} \cdot y_2^{(n)})}{d(x_1 \cdot x_2)} = 1.$$

Then

$$y_1^{(n)} = \phi_1(x_2),$$

$$y_2^{(n)} = \psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)},$$

ϕ_1, ψ_1 being arbitrary functions.

In just the same way we should get

$$y_1^{(n-1)} = \phi_2\{y_2^{(n)}\} = \phi_2\left\{\psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)}\right\},$$

$$y_2^{(n-1)} = \psi_2\{y_2^{(n)}\} - \frac{y_1^{(n)}}{\phi_2'(y_2^{(n)})}$$

$$= \psi_2\left\{\psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)}\right\} - \phi_1(x_2) : \phi_2'\left\{\psi_1(x_2) - \frac{x_1}{\phi_1'(x_2)}\right\}.$$

Continuing in this way, we shall get expressions of very great generality for y_1, y_2. But no solution thus obtained will be a general solution; nor can we even say that these solutions will include every possible solution, since we do not know that the forms attributed to y_1, say, include every possible form of function of x_1, x_2.

The manner of dealing with the more general equation

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = 1$$

is precisely similar. The results may be in certain cases extended to

the equation (24), when P is a product of a function of $y_1 \dots y_n$ by a function of $x_1 \dots x_n$. For we have

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = Y \cdot X \quad \dots\dots (28),$$

$$= \frac{d(y_1 \dots y_n)}{d(y'_1 \dots y'_n)} \cdot \frac{d(y'_1 \dots y'_n)}{d(y''_1 \dots y''_n)} \cdot \frac{d(y''_1 \dots y''_n)}{d(x_1 \dots x_n)}.$$

Now let $y''_1 \dots y''_n$ be particular solutions of

$$\frac{d(y''_1 \dots y''_n)}{d(x_1 \dots x_n)} = X;$$

and $y'_1 \dots y'_n$ be particular solutions of

$$\frac{d(y'_1 \dots y'_n)}{d(y_1 \dots y_n)} = \frac{1}{Y}.$$

Lastly, let us find as general a solution as possible of

$$\frac{d(y'_1 \dots y'_n)}{d(y''_1 \dots y''_n)} = 1.$$

By eliminating y', y'' from these results we shall get a solution of (28) of considerable generality.

We have already met with examples of systems of simultaneous binomial equations; for instance, (14) is such. These, however, in all cases in which their general solution has been attained, are included in the next class of equations. It will be convenient to defer the consideration of them.

Equations with Selected Terms.

14. The last class of equations with which we propose to deal only include certain selected terms of the general equation

$$\Sigma P \frac{d(y_1 \dots y_n)}{d(\xi_1 \dots \xi_n)} = 0 \quad \dots\dots (7).$$

These selected terms are distinguished by the circumstance that by replacing one of the independent variables in each Jacobian by one of the independent variables excluded from it, we arrive at one and the same Jacobian, which may or may not actually occur in the equation.

For convenience we will call the dependent variables $u_1 \dots u_n$; and will suppose the defining Jacobian of the equation to be

$$\frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)}.$$

The remaining independent variables we shall indicate by $x_1 \dots x_p$, so that there are in all $n+p$ independent variables. The general form

of the Jacobian which can occur in an equation of the class we now consider is

$$\frac{d(u_1 \dots u_i \dots u_n)}{d(y_1 \dots x_j \dots y_n)};$$

viz., *one* of the y's is replaced by *one* of the x's. The most general form of the equation itself is

$$P\frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)} + \Sigma P^i_j \frac{d(u_1 \dots u_i \dots u_n)}{d(y_1 \dots x_j \dots y_n)} = 0 \quad \dots\dots\dots (29).$$

A reference to (5) will show that

$$\frac{d(u_1 \dots u_i \dots u_n)}{d(y_1 \dots x_j \dots y_n)} : \frac{d(u_1 \dots u_n)}{d(y_1 \dots y_n)} = -\frac{dy_i}{dx_j}.$$

Thus (29) reduces to

$$\Sigma P^i_j \frac{dy_i}{dx_j} = P \ (i = 1 \dots n, \ j = 1 \dots m) \quad \dots\dots\dots (30),$$

which is a partial differential equation of the first order—involving only Jacobians of the first order.

If it should happen that all the i's in (29) or (30) were the same, then (30) would be a partial differential equation with *one* dependent variable. It is to this circumstance that the form of (15), for instance, is due.

Suppose, however, that the i's are different. Say we have r different y's and s different x's in (30): let these be $y_1 \dots y_r$, $x_1 \dots x_s$. Now choose r new dependent variables $\eta_1 \dots \eta_r$ functions of $y_1 \dots y_n$, $x_1 \dots x_m$, and m new independent variables $\xi_1 \dots \xi_m$ functions of $x_1 \dots x_m$, so that, if possible, (30) may reduce to the form

$$\frac{d\eta_1}{d\xi_1} + \frac{d\eta_2}{d\xi_2} + \dots + \frac{d\eta_r}{d\xi_r} = 0 \quad \dots\dots\dots (31).$$

The general solution of this equation is

$$\eta_1, \ -\eta_2, \ \dots, \ \pm\eta_r = \frac{d(\phi_1, \phi_2, \dots \phi_{r-1})}{d(\xi_1, \xi_2, \dots \xi_r)},$$

where $\phi_1, \dots \phi_{r-1}$ are $r-1$ arbitrary functions of $(\xi_1 \dots \xi_m)$ (see "Messenger of Mathematics," vol. vii., p. 107). This is also the general solution of (30), if after the differentiations $\eta_1 \dots \eta_r$, $\xi_1 \dots \xi_m$ be replaced by their values in terms of $x_1 \dots x_m$, $y_1 \dots y_n$. Also the solution of (29) may be written

$$\psi_k \{u_1, \dots, u_n, \ y_{r+1}, \dots, y_n, \ \eta_k (-)^k \Delta_k\} = 0 \quad \dots\dots\dots (32),$$
$$k = 1, \dots r,$$

where

$$\Delta_1, \Delta_2, \dots \Delta_r = \frac{d(\phi_1 \dots \phi_{r-1})}{d(\xi_1 \dots \xi_r)}.$$

For $u_1, u_2, \dots u_n = 0$ are solutions of (30).

A simple example of this class of equation is afforded by

$$\frac{d(u_1, u_2, u_3)}{d(x_1, x_2, x_3)} + \frac{d(u_1, u_2, u_3)}{d(x_3, x_4, x_5)} + \frac{d(u_1, u_2, u_3)}{d(x_2, x_4, x_6)}.$$

If we write

$$x_2 = \eta_1, \quad x_3 = \eta_2, \quad x_4 = \eta_3,$$
$$x_1 = \xi_1, \quad x_5 = \xi_2, \quad x_6 = \xi_3.$$

The above becomes

$$\frac{d(u_1, u_2, u_3)}{d(\eta_1, \eta_2, \xi_1)} + \frac{d(u_1, u_2, u_3)}{d(\xi_2, \eta_2, \eta_3)} - \frac{d(u_1, u_2, u_3)}{d(\eta_1, \xi_3, \eta_3)} = 0,$$

whence

$$\frac{d\eta_3}{d\xi_1} + \frac{d\eta_1}{d\xi_2} - \frac{d\eta_2}{d\xi_3} = 0,$$

or

$$\eta_3, -\eta_1, -\eta_2 = \frac{d(\phi_1, \phi_2)}{d(\xi_1, \xi_2, \xi_3)},$$

i.e.,

$$x_4, -x_2, -x_3 = \frac{d(\phi_1, \phi_2)}{d(x_1, x_5, x_6)} = \Delta_1, \Delta_5, \Delta_6 \text{ say},$$

where ϕ_1, ϕ_2 are arbitrary functions of x_1, x_5, x_6. The solution of the given equation is, therefore,

$$\psi_1\{u_1, u_2, u_3, x_4 - \Delta_1\} = 0,$$
$$\psi_2\{u_1, u_2, u_3, x_2 + \Delta_5\} = 0,$$
$$\psi_3\{u_1, u_2, u_3, x_3 + \Delta_6\} = 0.$$

15. There is no special difficulty as to simultaneous equations. As an example we may take the equations left in a previous article, viz.,

$$\frac{d(\eta_1, \eta_2)}{d(y_1, x_2)} = 0, \quad \frac{d(\eta_1, \eta_2)}{d(y_2, x_2)} + \frac{y_2}{x_1}\frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_1)} + \frac{d(\eta_1, \eta_2)}{d(y_1, x_3)} - \frac{y_2}{x_3}\cdot\frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, y_2)} + \frac{d(\eta_1, \eta_2)}{d(y_2, x_3)} - \left(\frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3}\right)\frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0,$$

$$\frac{d(\eta_1, \eta_2)}{d(x_1, x_2)} + \frac{d(\eta_1, \eta_2)}{d(x_2, x_1)} - \frac{y_2^2}{x_1 x_3}\cdot\frac{d(\eta_1, \eta_2)}{d(y_1, y_2)} = 0.$$

From these, taking y_1, y_2 as new dependent variables, and writing only independent equations,

$$\frac{dy_2}{dx_2} = 0, \qquad \frac{dy_2}{dx_1} - \frac{dy_2}{dx_3} - \frac{y_2}{x_3} = 0,$$

$$\frac{dy_1}{dx_2} + \frac{y_2}{x_1} = 0, \qquad \frac{dy_1}{dx_1} - \frac{dy_1}{dx_3} + \frac{y_1}{x_1} + \frac{x_2 y_2}{x_1 x_3} = 0.$$

From the first pair we get

$$y_2 = \frac{1}{x_3}\,\phi\,(x_1+x_3),$$

and then, from the second pair,

$$y_1 = -\frac{x_2}{x_1 x_3}\cdot\phi\,(x_1+x_3)+\frac{1}{x_1}\,\psi\,(x_1+x_3),$$

$$= -\frac{x_2}{x_1}\,y_2+\frac{1}{x_1}\,\psi\,(x_1+x_3).$$

These lead to the general solution found in Art. 7.

On the Relation between the Functions of Laplace and Bessel.

By Lord Rayleigh, M.A., F.R.S.

[*Read January* 10, 1878.]

In § 783 of Thomson and Tait's "Natural Philosophy" a suggestion is made to examine the transition from formulæ dealing with Laplace's spherical functions to the corresponding formulæ proper to a plane. It is evident at once from this point of view that Bessel's functions are merely particular cases of Laplace's more general functions; but the fact seems to be very little known. Of two valuable works recently published on this subject,* one makes no mention of Bessel's functions, and the other states expressly that they are not connected with the main subject of the book; other mathematicians also, to whom I have mentioned the matter, have been unaware of the relation. Under these circumstances it may not be superfluous to point out briefly the correspondence of some of the formulæ.

The Bessel's function of zero order J_0 is the limiting form of Legendre's function $P_n(\mu)$, when n is indefinitely great, and μ $(=\cos\theta)$ such that $n\sin\theta$ is finite, equal say to z. The simplest proof of this assertion is perhaps that obtained from Murphy's series for P_n. Thus (Todhunter, § 23),

$$P_n(\cos\theta) = \cos^{2n}\frac{\theta}{2}\left\{1-\binom{n}{1}\tan\frac{\theta}{2}\right)^2+\left(\frac{n\,.\,n-1}{1\,.\,2}\tan^2\frac{\theta}{2}\right)^2-\ldots\ldots\Bigg\} \qquad \ldots\ldots\ldots\ldots(1).$$

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* "An Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions." By I. Todhunter. 1875.

"An Elementary Treatise on Spherical Harmonics and subjects connected with them." By the Rev. N. M. Ferrers. 1877.

The limit of $\cos^{2n}(\frac{1}{2}\theta)$ is easily proved to be unity, and thus ultimately

$$P_n(\cos\theta) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2.4^2} - \frac{z^6}{2^2.4^2.6^2} + \ldots\ldots = J_0(z) \ldots\ldots (2).$$

Again (§ 47),

$$\pi P_n(\mu) = \int_0^\pi \{\mu + i\sqrt{(1-\mu^2)}\cos\phi\}^n d\phi \ldots\ldots\ldots\ldots (3),$$

where $i = \sqrt{(-1)}$. In the limit μ is to be replaced by unity, and $\sqrt{(1-\mu^2)}$ by zn^{-1}; the limit of $\{\mu + i\sqrt{(1-\mu^2)}\cos\phi\}^n$ is $e^{iz\cos\phi}$, and thus we get ultimately

$$\pi P_n(\mu) = \int_0^\pi e^{iz\cos\phi}\, d\phi = \int_0^\pi \cos(z\cos\phi)\, d\phi = \pi J_0(z).$$

In like manner Bessel's functions of higher order are the limits of those Laplace's functions called by Todhunter Associated Functions, which, when multiplied by sines or cosines of multiples of the longitude (ω), satisfy Laplace's spherical surface equation,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_n}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2Y_n}{d\omega^2} + n(n+1)\,Y_n = 0 \ldots\ldots (4).$$

They may be expressed (§§ 99, 106) by

$$(1-\mu^2)^{\frac{1}{2}m}\,\varpi(m, n, \cos\theta) \ldots\ldots (5),$$

where

$$\varpi(m, n, \cos\theta)$$
$$= \varpi(m, n, 1)\left\{\cos^{n-m}\theta - \frac{(n-m)(n-m-1)}{4.(m+1).1}\cos^{n-m-2}\theta\sin^2\theta\right.$$
$$\left. + \frac{(n-m)\ldots(n-m-3)}{4^2.(m+1)(m+2).1.2}\cos^{n-m-4}\theta\sin^4\theta - \ldots\ldots\right\} \ldots\ldots (6).*$$

Now $(1-\mu^2)^{\frac{1}{2}m}$ is proportional to z^m; and the function becomes

$$z^m\left\{1 - \frac{z^2}{2(2m+2)} + \frac{z^4}{2.4.(2m+2)(2m+4)} - \ldots\ldots\right\},$$

which is proportional to the Bessel's function of order m (§ 370).

The same conclusion may be arrived at from the expression for the associated functions as definite integrals. Thus (§ 150) c_m^n is an associated function,† where

$$c_m^n = \int_0^\pi \{\mu + i\sqrt{(1-\mu^2)}\cos\phi\}^n \cos m\phi\, d\phi \ldots\ldots\ldots\ldots (7).$$

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* This series is not quite correctly given in § 106.

† In Heine's and Todhunter's works this integral is called J_n, a notation which will be seen to be inappropriate.

By the investigation in § 150, we have

$$c_m^n = \frac{(-1)^{\frac{1}{2}m} \cdot 1 \cdot 2 \cdot 3 \ldots n \cdot (1-\mu^2)^{\frac{1}{2}m}}{1 \cdot 3 \cdot 5 \ldots (2m-1) \cdot 1 \cdot 2 \cdot 3 \ldots (n-m)} \times \int_0^\pi \{\mu + i\sqrt{(1-\mu^2)}\cos\phi\}^{n-m} \sin^{2m}\phi \, d\phi \quad \ldots\ldots (8),$$

from which we find, as for the function of the first order, when $n=\infty$, $n\sqrt{(1-\mu^2)} = z$,

$$c_m^\infty = (-1)^{\frac{1}{2}m} \pi J_m(z) \quad \ldots\ldots (9).$$

It is known (§ 380) that the following relation subsists between three consecutive Bessel's functions:

$$\tfrac{1}{2}z\{J_{m-1}(z) + J_{m+1}(z)\} = m J_m(z) \quad \ldots\ldots (10);$$

and it is natural to suppose that a corresponding theorem holds for the general functions. By (7),

$$\tfrac{1}{2}(c_{m-1}^n - c_{m+1}^n) = \int_0^\pi \{\mu + i\sqrt{(1-\mu^2)}\cos\phi\}^n \sin m\phi \sin\phi \, d\phi,$$

whence, integrating by parts, we find (m being integral)

$$\tfrac{1}{2}(c_{m-1}^n - c_{m+1}^n) = \frac{m}{i\sqrt{(1-\mu^2)} \cdot (n+1)} c_m^{n+1} \quad \ldots\ldots (11);$$

and this, as we see from (9), identifies itself with (10), when $n=\infty$.

Again, by (7),

$$\tfrac{1}{2}(c_{m-1}^n + c_{m+1}^n) = \int_0^\pi \{\mu + i\sqrt{(1-\mu^2)}\cos\phi\}^n \cos m\phi \cos\phi \, d\phi \ldots (12);$$

and thus

$$c_m^{n+1} = \mu c_m^n + \tfrac{1}{2}i\sqrt{1-\mu^2}\,(c_{m-1}^n + c_{m+1}^n) \quad \ldots\ldots (13).$$

From (11) and (13) we may eliminate c_m^{n+1}; we thus obtain

$$2m c_m^n = i \tan\theta \{(n+1-m) c_{m-1}^n - (n+1+m) c_{m+1}^n\} \quad \ldots\ldots (14),$$

a relation given in a somewhat different form by Heine.*

Returning to (12), we find in the limit, by (9),

$$\tfrac{1}{2}(-1)^{\frac{1}{2}m}\pi\{-i J_{m-1}(z) + i J_{m+1}(z)\} = \int_0^\pi e^{iz\cos\phi} \cos m\phi \cos\phi \, d\phi = -i\frac{d}{dz}\int_0^\pi e^{iz\cos\phi}\cos m\phi \, d\phi;$$

so that

$$\tfrac{1}{2}\{J_{m-1}(z) - J_{m+1}(z)\} = \frac{d}{dz} J_m(z) \quad \ldots\ldots (15),$$

a known theorem (§ 381).

The complete solution of the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\psi_m}{d\mu}\right\} - \frac{m^2}{1-\mu^2}\psi_m + n(n+1)\psi_m = 0 \quad \ldots\ldots (16),$$

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* "Handbuch der Kugelfunctionen," p. 146.

obtained from (4) by assuming that ψ_m is proportional to $\cos m\omega$ or $\sin m\omega$, involves also another function called the associated function of the second kind. If we take

$$\psi_m = (1-\mu^2)^{\frac{1}{2}m}\,\phi_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

the equation for ϕ_m is

$$(1-\mu^2)\frac{d^2\phi_m}{d\mu^2} - 2(m+1)\mu\frac{d\phi_m}{d\mu} + \left\{n(n+1) - m(m+1)\right\}\phi_m = 0 \ldots (18).$$

From this it may be readily proved that ϕ_m is of the same form as $\frac{d}{d\mu}\phi_{m-1}$, so that

$$\phi_m = \left(\frac{d}{d\mu}\right)^m \phi_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

in which, however, no connection between the arbitrary constants is implied. In terms of ψ_m, (19) becomes

$$\psi_m = (1-\mu^2)^{\frac{1}{2}m}\left(\frac{d}{d\mu}\right)^m \psi_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20).$$

We have now to trace the transformation of (20), as n and the radius of the sphere become infinite, while $n\nu = n\sqrt{(1-\mu^2)} = z$.

Since $\mu^2+\nu^2 = 1$, $\qquad \psi_m = \nu^m\left(-\frac{\mu}{\nu}\frac{d}{d\nu}\right)^m \psi_0$,

ψ_0 being regarded as a function of ν.

In the limit, μ (even though subject to differentiation) may be identified with unity, and thus we may take

$$\psi_m = -(2z)^m\left(\frac{d}{d\,.\,z^2}\right)^m \psi_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

as the ultimate form of (20). The function ψ_m is now the general solution of

$$\frac{d^2\psi_m}{dz^2} + \frac{1}{z}\frac{d\psi_m}{dz} + \left(1-\frac{m^2}{z^2}\right)\psi_m = 0 \quad \ldots\ldots\ldots\ldots\ldots (22),$$

involving Bessel's functions of order m *of both kinds*, each function being affected with an arbitrary coefficient. A theorem of precisely the same form holds good when ψ_m is limited to the functions of the first kind, for which the differential equations (16), (22) are satisfied at the pole ($\mu = 1$) as well as at all other points. The equation corresponding to (21) may then be written (§ 390)

$$J_m(z) = (-2z)^m\left(\frac{d}{d\,.\,z^2}\right)^m J_0(z) \quad \ldots\ldots\ldots\ldots\ldots (23),$$

the constant multiplier left arbitrary by (21) being determined by a comparison of the leading terms.

Other comparisons relating to the functions of the second kind might also be made, some of which would appear to involve theorems not hitherto formulated.

Notes on the Normals of Conics. By Samuel Roberts, M.A.

[*Read December* 13, 1877.]

On a subject so well explored I do not venture to assert that my results are new. They are obtained by simple methods, and it may be of some use if I bring together in a more or less consecutive way theorems scattered in various repertories. At the same time, I am inclined to think that a few of my conclusions are, if not altogether novel, put in a somewhat new light.

I.

1. The poles of a rectangular pair of conjugate straight lines, with respect to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, are connected by the equations

$$x_1 x_2 = \frac{a^4}{c^2}, \quad y_1 y_2 = -\frac{b^4}{c^2}.$$

If, therefore, one pole describes the curve $F(x, y) = 0$, the contrapole describes the curve

$$F\left(\frac{a^4}{c^2 x}, -\frac{b^4}{c^2 y}\right) = 0.$$

In this and similar cases it seems unnecessary to specify characteristics which may be derived from the theory of inversion and substitution according to known rules.

If the lines $\alpha x+\beta y+\gamma=0$ ... (1), $\alpha' x+\beta' y+\gamma'=0$ ... (2) are rectangular conjugates with respect to the same conic, we have

$$\frac{\alpha\alpha'}{\gamma\gamma'} = \frac{1}{c^2}, \qquad \frac{\beta\beta'}{\gamma\gamma'} = -\frac{1}{c^2}.$$

When the line (1) envelopes the class curve $F(\alpha, \beta, \gamma) = 0$, the line (2) envelopes the class curve $F\left(\frac{1}{\alpha}, -\frac{1}{\beta}, \frac{c^2}{\gamma}\right) = 0$.

It follows from the forms of the expressions that confocal conics have the same rectangular conjugates.

If (x_1, y_1), the pole of (1), describes the curve $F(x_1, y_1, 1) = 0$, the normal conjugate (2) envelopes the class curve

$$F\left(\frac{a^2}{\alpha}, -\frac{b^2}{\beta}, -\frac{c^2}{\gamma}\right) = 0.$$

All the polars of a point (x_1, y_1) with respect to a system of concentric, coaxial, and similar conics have the same normal conjugate.

In virtue of the relation of pole and polar, if (x_1, y_1) describes the curve $F(x, y, 1) = 0$, the line (1) will envelope the class curve $F(a^2\alpha, b^2\beta, -\gamma) = 0$.

2. In particular, if the line (1) envelope the point $A\alpha+B\beta+C\gamma=0$, the normal conjugate (2) will envelope the parabola

$$\frac{A}{\alpha}-\frac{B}{\beta}+\frac{c^2C}{\gamma}=0$$

(Chasles' "Coniques," p. 145), which remains the same for a confocal system of conics.

Considering c^2 as a variable parameter, we have a system of conics with three common tangents and a common point of contact, or four common tangents, two of which are consecutive. The point-pair $\alpha\beta=0$ consists of the poles of the axes. The other point-pair $\gamma(A\beta-B\alpha)=0$ consists of the common centre and the point at infinity of the parabolas.

For a given confocal system, the common tangents of the parabola and the conics of the system determine, at the points of contact with a particular curve, the feet of normals concurrent at the given point.

The curve touches the polars of the point with respect to the system, that is to say, the equation is satisfied by $\left(\frac{A}{a^2+k}, \frac{B}{b^2+k}, -C\right)$.

This envelope, which is well known to be a parabola, has therefore for its Cartesian equation

$$(xx_1+yy_1-a^2-b^2)^2+4(a^2yy_1+b^2xx_1-a^2b^2)=0,$$

obtained in the usual way from

$$\frac{x_1x}{a^2+k}+\frac{y_1y}{b^2+}-1=0.$$

The equation may be written

$$(x_1y+y_1x)^2+(xx_1-yy_1-c^2)^2-(xy_1-x_1y)^2=0,$$

showing the focus and the directrix which passes through the centre.

The parabola can readily be drawn by points, for it touches the axis of x, where the polar of the given point (x_1, y_1) with respect to the concentric circle through the real foci meets the axis; and it meets the axis of y in a point similarly determined by the polar of the point $(-x_1, -y_1)$ relative to the same circle. This conic seems to be as interesting in its geometrical relations as the perhaps more familiar hyperbola

$$-\frac{Aa^2}{x}+\frac{Bb^2}{y}+Cc^2=0,$$

which remains the same for a similar system of conics, and passes through the given point. It is the locus of the pole of the normal conjugate (2), that is to say, the reciprocal of the parabola relative to the given conic.

3. In like manner, if (x_1, y_1), the pole of (1), describes the line $Ax+By+C=0$, the normal conjugate (2) envelopes the parabola

$\frac{Aa^2}{\alpha} - \frac{Bb^2}{\beta} - \frac{Cc^2}{\gamma} = 0$, which remains the same for a system of concentric, coaxial, and similar conics.

This parabola and the corresponding hyperbola, $\frac{Aa^4}{c^2x} - \frac{Bb^4}{c^2y} + C = 0$, are the same as before, but arranged in different systems.

4. Two rectangular conjugates, with the polar of their intersection or apex, constitute a right-angled self-conjugate triangle. The coordinates of the apex are

$$X = \frac{b^4 + c^2y_1^2}{a^4y_1^2 + b^4x_1^2}\, a^2x_1 = \frac{b^4 + c^2y_2^2}{a^4y_2^2 + b^4x_2^2}\, a^2x_2,$$

$$Y = \frac{a^4 - c^2x_1^2}{a^4y_1^2 + b^4x_1^2}\, b^2y_1 = \frac{a^4 - c^2x_2^2}{a^4y_2^2 + b^4x_2^2}\, b^2y_2.$$

We can, therefore, write down the equation of the locus of the poles when that of (X, Y) is given.

If, as before, $\alpha x + \beta y + \gamma = 0$ ...(1), $\alpha' x + \beta' y + \gamma' = 0$ ...(2) are the equations of the rectangular conjugates, the coordinates of the apex are

$$X = \frac{\alpha(\gamma^2 + c^2\beta^2)}{-\gamma(\alpha^2 + \beta^2)} = \frac{\alpha'(\gamma'^2 + c^2\beta'^2)}{-\gamma'(\alpha'^2 + \beta'^2)},$$

$$Y = \frac{\beta(\gamma^2 - c^2\alpha^2)}{-\gamma(\alpha^2 + \beta^2)} = \frac{\beta'(\gamma'^2 - c^2\alpha'^2)}{-\gamma'(\alpha'^2 + \beta'^2)}.$$

Thus, if $Y=0$, the conjugates envelope the foci $\gamma \pm c\alpha = 0$, *i. e.*, we have the theorem of De la Hire. The locus of each of the poles is a directrix. The point $\beta=0$ is the pole of the major axis.

The equation of the hypothenuse of the triangle, being the polar of the apex, is

$$X\frac{\alpha}{a^2}(\gamma^2 + c^2\beta^2) + Y\frac{\beta}{b^2}(\gamma^2 - c^2\alpha^2) + \gamma(\alpha^2 + \beta^2) = 0;$$

so that in general, if the conjugates envelope a curve of class n, the hypothenuse envelopes a curve of class $3n$. This class is, however, reduced for special forms. If a conjugate envelopes a point $\beta = k\gamma$ on the minor axis, the envelope of the hypothenuse is a conic. The other conjugate also envelopes a point on the minor axis, and the apex describes a circle.

In fact, if the coordinates of the apex (X, Y) are given, we get, to determine (x_1, y_1), (x_2, y_2), the equations

$$x^2 - \frac{a^2x}{c^2X}(X^2 + Y^2 + c^2) + \frac{a^4}{c^2} = 0,$$

$$y^2 + \frac{b^2y}{c^2Y}(X^2 + Y^2 - c^2) - \frac{b^4}{c^2} = 0.$$

We can therefore write down the equation of the locus of the apex, when the locus of the middle point of the hypothenuse is given.

Thus, if $\frac{1}{2}(y_1+y_2)=k$, which also implies that (x_1, y_1), (x_2, y_2) move on lines parallel to the major axis, the locus of X, Y is a circle, viz., we have $$b^2(X^2+Y^2-c^2)+2c^2kY=0.$$

The two normal conjugates envelope the extremities of the diameter coincident with the minor axis. In general, if the locus of the middle point is a line, the locus of (X, Y) is a circular cubic. The locus of the apex is, from the nature of the case, the locus of the apex of a right angle whose sides envelope certain curves. If one of the conjugates envelopes a point, the locus of the apex is a pedal of the parabola enveloped by the normal conjugate.

5. When the pole (x_1, y_1) of (1) moves on a concentric and coaxial conic, $$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}-1=0,$$

the normal conjugate (2) envelopes the curve

$$\frac{a^4}{a_1^2\alpha^2}+\frac{b^4}{b_1^2\beta^2}-\frac{c^4}{\gamma^2}=0.$$

This is the tangential equation of the evolute of

$$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}-1=0,$$

provided $$a_2^2=\frac{a_1^2b_1^4a^4c^4}{(a^4b_1^2-b^4a_1^2)^2}, \qquad b_2^2=\frac{a_1^4b_1b^4c^4}{(a^4b_1^2-b^4a_1^2)^2}.$$

Hence, if the pole of a line moves on a concentric and coaxial conic, the normal conjugates are normals to a third concentric and coaxial conic.

We have also $$a_1^2=\frac{c_2^4a^4}{c^4a_2^2}, \qquad b_1^2=\frac{c_2^4b^4}{c^4b_2^2}.$$

Hence the poles of the normal conjugates of the normals of a conic

$$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}-1=0,$$

with respect to the conic $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

lie on the conic $$\frac{c^4a_2^2}{c_2^4a^4}x^2+\frac{c^4b_2^2}{c_2^4b^4}y^2-1=0;$$

and the poles of the normals themselves with respect to the original conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ lie on the curve

$$\frac{a_2^2a^4}{c_2^4x^2}+\frac{b_2^2b^4}{c_2^4y^2}-1=0.$$

6. In like manner, if $\alpha x+\beta y+\gamma=0$ ...(1) envelopes the conic whose tangential equation is

$$a_1^2\alpha^2+b_1^2\beta^2-\gamma^2=0,$$

the corresponding normal conjugates will envelope

$$\frac{a_1^2}{\alpha^2}+\frac{b_1^2}{\beta^2}-\frac{c^4}{\gamma^2}=0,$$

which is the evolute of $\dfrac{x^2}{a_2^2}+\dfrac{y^2}{b_2^2}-1=0,$

provided $a_2^2=\dfrac{c^4}{c_1^4}\,a_1^2, \quad b_2^2=\dfrac{c^4}{c_1^4}\,b_1^2;$

the normal conjugates of (1) are therefore normals of a third concentric and coaxial conic.

Without departing from the relation of special quadric inversion, we might take the polar of a point (x_1, y_1) with regard to

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}-1=0,$$

and its normal conjugate with regard to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

The poles of the lines with respect to the first conic are connected by

$$x_1x_2=\frac{a_1^4}{c^2}, \qquad y_1y_2=-\frac{b_1^4}{c^2}.$$ *

II.

7. If p is the perpendicular from the centre to the normal of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

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* I have been referred to a paper "On Normals to Conics: a New Treatment of the Subject," by Professor Cremona ("Messenger of Mathematics," Vol. III., pp. 88–91, 1864). The author, in his usual luminous way, applies the quadric transformation in its geometrical form, when lines correspond to lines and points to conics. The general correspondence and its special case are thus stated:—"Let aa', bb', cc' be the vertices of a quadrilateral whose diagonals aa', bb', cc' form a triangle $\alpha\beta\gamma$. Any line R intersects the diagonals in three points whose harmonic conjugates relative to the couples aa', bb', cc' respectively lie on another line R' which may be said to correspond to R." "When R turns round a fixed point p, R' envelopes a conic P inscribed in $\alpha\beta\gamma$." And then, "if the points c, c' coincide with the imaginary circular points at infinity, the inscribed conics will form a system of confocal conics: a, a' and b, b' being their common foci (real and imaginary), and γ their common centre." "Corresponding lines R, R' are now perpendicular to each other, and divide harmonically the focal segments aa', bb'."

What I have given is merely an analytical representation of this theory, and, as such, is perhaps worth retaining.

we have $$p^2 = \frac{c^4 \sin^2\theta \cos^2\theta}{a^2\cos^2\theta + b^2\sin^2\theta},$$

where θ is the angle which the normal makes with the axis of x. The equation, in fact, represents the pedal of the evolute with regard to the centre. If, then, $xM - yL = 0$ is a normal referred to axes parallel to the principal axes of the conic, and X, Y are the coordinates of the centre, $$(XM - YL)^2 = \frac{c^4 L^2 M^2}{a^2L^2 + b^2M^2}.$$

We may interpret this result in several ways. If X, Y are considered as current coordinates, and the origin is supposed to be at the centre, the equation represents two parallel normals. If X, Y are variable coordinates of the centre, two parallel lines are represented along which the centre of the conic must move, while the given line $xM - yL = 0$ remains normal, and the conic is simply translated. If we take L, M as variable parameters, the equation represents (1) the four normals which can be drawn to the conic for a given position of its centre; (2) the four normals which can be drawn to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ through a point (X, Y).

The expanded form is

$$a^2Y^2L^4 - 2a^2XYL^3M + (b^2Y^2 + a^2X^2 - c^4)\,L^2M^2 - 2b^2XYLM^3 + b^2X^2M^4 = 0.$$

For the parabola $y^2 = px$ we get a corresponding equation,

$$M^3p + 2(p - 2X)\,L^2M + 4L^3Y = 0.$$

8. If m_1, m_2, m_3, m_4, μ are the tangents of the angles which the four normals through (X, Y) and the line joining (X, Y) to the centre make with the axis of x,

$$m_1 + m_2 + m_3 + m_4 = 2\mu, \qquad \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} = \frac{2}{\mu}.$$

The quadratic and cubic invariants are

$$S = \tfrac{1}{12}(a^2x^2 + b^2y^2 - c^4)^2,$$
$$T = -\tfrac{1}{216}\{(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2c^4x^2y^2z^2\}.$$

The conic $a^2x^2 + b^2y^2 - c^4 = 0$ is the locus of points normals from which form an equi-anharmonic pencil, *i. e.*, the value of the anharmonic ratio is one of the imaginary cube roots of negative unity. The conic is the reciprocal of the given conic with respect to either concentric circle through two foci. In like manner, $T = 0$ (J. J. Walker, "Educational Times," April, 1871) is the locus of points from which the normals constitute an harmonic pencil, and generally for a constant anharmonic ratio $T^2 - k^2S^3 = 0$, or $T \pm kS^{\frac{3}{2}} = 0$, a locus consisting of two curves of the sixth degree. In the case of the discriminant, these

become the evolute and the axes and line at infinity twice over. For shortness, I sometimes call the conic $S^{\frac{1}{3}} = 0$ the conic of the cusps. In the case of the evolute, it is touched by the eight tangents at the four double points. Several relations noticed by M. Laguerre ("Comptes Rendus," Janvier, 1877) may be immediately obtained from the foregoing equation of normals.

9. Consider $\alpha x+\beta y+\gamma = 0$ as a chord of the conic

$$F(x, y) = a'x^2+b'y^2+c'+2f'y+2g'x+2h'xy = 0.$$

To obtain the equation of the normal conjugates relative to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = 0$ of the polars of the intersections of the chord with respect to the latter conic, we have

$$F(x_1, y_1) = 0,$$

$$\alpha x_1+\beta y_1+\gamma = 0,$$

$$\frac{a^2X}{c^2x_1}-\frac{b^2Y}{c^2y_1}-1 = 0.$$

Let the elimination of y_1 from the first and second of these equations give $$Lx_1^2+Mx_1+N = 0.$$

By the second and third of the equations,

$$c^2\alpha x_1^2-Px_1-a^2\gamma X = 0,$$

where P is written for $a^2\alpha X+b^2\beta Y-c^2\gamma$.

The required equation therefore is

$$U = (c^2\alpha N+a^2\gamma LX)^2+(PL+c^2\alpha M)(PN-a^2\gamma MX) = 0.$$

We can get the coordinates of the intersection of these most readily by making $\frac{dU}{dY} = 0$, $\frac{dU}{dX} = 0$.

Thus we find

$$2NLP-a^2\gamma XML+c^2\alpha MN = 0,$$

$$-MLP+2a^2\gamma L^2X+c^2\alpha(2LN-M^2) = 0,$$

giving $$a^2\gamma LX+c^2\alpha N = 0.$$

The Y coordinate is obtained by symmetry, and we have

$$X = -\frac{c^2\alpha(b'\gamma^2+c'\beta^2-2f'\beta\gamma)}{a^2\gamma(a'\beta^2+b'\alpha^2-2h'\alpha\beta)},$$

$$Y = \frac{c^2\beta(a'\gamma^2+c'\alpha^2-2g'\alpha\gamma)}{a^2\gamma(a'\beta^2+b'\alpha^2-2h'\alpha\beta)}.$$

Thus, when the locus of (X, Y) is given, we can write down the tangential equation of the envelope of the chord.

10. In particular, if the conic $F(x, y)$ is identical with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

we have $$X=-\frac{c^2\alpha\left(\frac{\gamma^2}{b^2}-\beta^2\right)}{a^2\gamma\left(\frac{\beta^2}{a^2}+\frac{\alpha^2}{b^2}\right)}, \qquad Y=\frac{c^2\beta\left(\frac{\gamma^2}{a^2}-\alpha^2\right)}{b^2\gamma\left(\frac{\beta^2}{a^2}+\frac{\alpha^2}{b^2}\right)}.$$

If the intersection of the normals at the extremities of the chord moves on a line, $Ax+By+C=0$,

the chord envelopes the class cubic

$$-c^2\alpha\,(\gamma^2-\beta^2b^2)\,A+c^2\beta\,(\gamma^2-\alpha^2a^2)\,B+\gamma\,(\beta^2b^2+\alpha^2a^2)\,C=0.$$

It is easy to see, from the form of the equation and *à priori*, the principal specialities of the curve, which has been found geometrically and described at some length by Dr. Emil Weyr ("Schlömilch," Z. xvi., 440-4). He has remarked that, if the given line is a normal, the above curve breaks up into a parabola and the foot of the normal (A). In fact, by comparing the equation with

$$(p\alpha+q\beta+r\gamma)\,(t\beta\gamma+u\alpha\gamma+v\alpha\beta),$$

the equation of the parabola is obtained in the form

$$\frac{1}{c^2A\alpha}-\frac{1}{c^2B\beta}+\frac{1}{C\gamma}=0,$$

with the linear factor $$-\frac{a^2\alpha}{A}+\frac{b^2\beta}{B}+\frac{c^2\gamma}{C}=0,$$

the tangential equation of the foot of the fixed normal.

The Cartesian equation of the parabola in one form is

$$\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+1\right)^2-\frac{4x_1y_1xy}{a^2b^2}=0,$$

where (x_1, y_1) is the foot of the fixed normal. The focus is on the tangent at $(-x_1, -y_1)$. The directrix passes through the centre, and is parallel to the normal at $(-y_1, x_1)$. The curve can be readily constructed by points. The locus of foci of the parabolas is the central pedal of the conic (B).*

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* In a paper "On Normals to Conics and Quadrics," ("Quarterly Journal of Mathematics," Vol. viii., p. 66, 1866,) Mr. Purser has given a number of interesting theorems, depending, so far as conics are concerned, on an identity which in its homogeneous form is

$$Ax^2+By^2+Cz^2+k\,(axy+byz+czx)$$
$$=(\alpha x+\beta y+\gamma z)\,(\alpha' x+\beta' y+\gamma' z),$$

giving $\alpha\alpha'=A, \quad \beta\beta'=B, \quad \gamma\gamma'=C.$

This the Author applies to the chords joining the feet of concurrent normals. Mr. Purser's results include (A) and (B).

11. Taking now, as previously, (x_1, y_1) for the pole of the chord, we have

$$\alpha : \beta : \gamma :: \frac{x_1}{a^2} : \frac{y_1}{b^2} : -1.$$

Consequently, if the intersection of the normals moves on the line $Ax+By+C=0$, the pole of the chord describes the cubic

$$-\frac{c^2x}{a^2}\left(1-\frac{y^2}{b^2}\right)A+\frac{c^2y}{b^2}\left(1-\frac{x^2}{a^2}\right)B-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)C=0,$$

or, if $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=U$,

$$\frac{c^2x^3}{a^4}A-\frac{c^2y^3}{b^4}B+C-U\left(\frac{c^2A}{a^2}x-\frac{c^2B}{b^2}y-C\right)=0.$$

We know, by simple substitution of the coordinates of a centre of curvature, that $\frac{c^2x^3}{a^4}A-\frac{c^2y^3}{b^4}B+C=0$ gives, by its intersections with the conic, six points whose centres of curvature range on the line $Ax+By+C=0$.

12. If this directrix is a normal, we have, as before, a compound locus consisting of the hyperbola

$$\frac{b^2x_1}{B}-\frac{a^2y_1}{A}+\frac{c^2x_1y_1}{C}=0,$$

and the line factor

$$-\frac{x_1}{A}+\frac{y_1}{B}-\frac{c^2}{C},$$

representing the tangent at the foot of the fixed normal.

The intersections, then, of the above hyperbola with the conic determine points whose centres of curvature lie on a normal, *i. e.*, they are the four simple intersections of a normal with the evolute.

When we write $\frac{x}{x_2}+\frac{y}{y_2}+1$ for $Ax+By+C=0$, the hyperbola in question becomes identical with that which determines the feet of normals concurrent at (x_2, y_2).

Since the directrix line is also a normal, we have the further condition,

$$a^2x_2^2+b^2y_2^2-c^4=0;$$

that is to say, the four simple intersections of a normal with the evolute are the centres of curvature of points the normals at which meet on the conic of the cusps.

The point (x_2, y_2) is formed by taking the intercepts of the directrix normal on the axes from the centre in a sense of opposite sign for the coordinates of the point. We thus have a construction by which, for a given normal, we can find the two real intersections with the evolute.

We have to draw normals from (x_2, y_2). Their intersections with the given normal are the points required, which are at the same time the centres of curvature of the curve at their feet. There are some interesting details with regard to this construction which I pass over.

13. In the case of the parabola, the conic of the cusps degenerates into the principal ordinate through the centre of curvature of the vertex and the line at infinity. To adapt the construction to this case, we may make use of the angles with the principal axis formed by the given normal and a right line joining (x_2, y_2) and the intersection of the given normal with the axis. Let ϕ be the first angle, and θ the second; then $\tan \phi = -2 \tan \theta$. Suppose that a normal p is drawn to a parabola. Another normal q, having its angle θ with the axis determined by the foregoing equation (exhibiting a simple geometrical relation), will intersect the first normal p at the centre of curvature belonging to the foot of the secondary normal q. This construction corresponds to an infinitely distant point (x_2, y_2). But a normal will intersect the principal ordinate through the centre of curvature of the vertex (being the other line of the degenerate conic of the cusps) in a finite point. In connection with this intersection, the intercept on the normal between the ordinate and the axis is one-half of the intercept between the axis and the centre of curvature. It will be seen now that this is the construction given immediately by the equation of normals. For a parabola we have $m_1+m_2+m_3 = 0$, and, making $m_2 = m_3$, we get $m_1 = -2m_2$.

In the case of a central conic, we have, making $m_3 = m_4$,

$$\tan \chi = \frac{m_1+m_2}{1-m_1m_2} = 2\,\frac{\mu-m_3}{1+m_3\mu} = 2 \tan \psi,$$

which is useful in some constructions. It thus appears that a normal to the conic intercepts the conic of the cusps in two points, whose coordinates, with their signs changed, determine the two normals which can be drawn from the centre of curvature of the foot of the first normal. We are thus led back to a tangential equation of the evolute,

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}-c^4 = 0,$$

where x, y are now considered as tangential coordinates.

14. The four centres of curvature on normals meeting at a point of the conic of the cusps lie on a linear transversal, and form an equi-anharmonic range, since the four normals or tangents of the evolute form an equi-anharmonic pencil.

Again, with regard to the reciprocal quartic, if four tangents be drawn from any point on the curve, they form an equi-anharmonic

pencil. Moreover, they meet the quartic again in points on a line which form an equi-anharmonic range. It is a speciality of the reciprocal unicursal quartic that the tangents at the nodes are harmonic conjugates to the adjacent nodal chords, a property which carries with it other special singularities, and cannot be altered by projection. We are, however, entitled to say that a unicursal quartic possessing those singularities is such that tangents from a point on it form a pencil whose anharmonic ratio is constant, and this is of some interest in connection with the well-known corresponding property of cubic curves.

February 14th, 1878.

Lord RAYLEIGH, F.R.S, President, in the Chair, and subsequently C. W. MERRIFIELD, Esq., F.R.S., Vice-President.

The following communications were made to the Society:—

"On a General Method of Solving Partial Differential Equations:" Prof. H. W. Lloyd Tanner.

"On the Conditions for Steady Motion of a Fluid:" Prof. H. Lamb, Adelaide.

"On a property of the Four-piece Linkage," and "On a curious Locus in Linkages:" Mr. A. B. Kempe.

"On Robert Flower's new Mode of Computing Logarithms:" Mr. S. M. Drach.

"On the Plückerian Characteristics of the Modular Equations:" Prof. H. J. S. Smith, Vice-President.

Mr. Drach also exhibited a large collection of drawings of Tricircloids made many years since for Mr. Perigal.

The following presents were received:—

"Nouveau Mode de Représentation plane de Classes de Surfaces régleés" (29 Oct. 1877).

"Applications d'un Mode de Représentation plane de Classes de Surfaces régleés" (5 Nov. 1877).

"Nouvelles applications d'un Mode de Représentation plane de Classes de Surfaces réglées" (19 Nov. 1877): par M. Mannheim.

Notes de M. Mannheim:—Première Note, "Sur le Déplacement infiniment petit d'un Dièdre de grandeur invariable" (11 Juin 1877): Seconde Note, "Sur les Courbes ayant les mêmes Normales principales, et sur la Surface formée par ces Normales" (23 Juillet 1877)—from the "Comptes Rendus." From the Author.

"Mémoire de l'Equilibre Astatique, et sur l'effet que peuvent produire des forces de grandeurs et de directions constantes appliquées en des points déterminés d'un corps solide quand ce corps change de position dans l'espace," par G. Darboux; Bordeaux, 1877.

"Educational Times," March 1878.

"Monatsbericht," Nov. 1878; Berlin, 1878.

"Annali di Matematica," Serie ii^a, Tomo viii°, Fasc. 4°, Dicembre 1877.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2^e Série, Tome ii, 2^e Cahier; Paris, 1878.

"Atti della R. Accademia dei Lincei . . . ." Serie terza; Transunti, Vol. ii, Fasc. 1°, Dicembre 1877; Fasc. 2°, Gennaio, 1878; Roma, 1878.

On a General Method of Solving Partial Differential Equations.

By H. W. Lloyd Tanner, M.A.

[*Read February* 14, 1878.]

It is well known that the general method of solving a partial differential equation of the first order consists in forming n equations, including the given one, such as to render

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

an integrable equation. These equations are found by solving certain partial differential equations usually written, for short,

$$[F_i, F_j] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

The object of the following paper is to deduce a system equivalent to (b) directly from the conditions of integrability of (a). Such a system is, in fact, obtained, and precisely the same form of equation serves to integrate equations of the higher orders.

It is shown that only one system of the kind just mentioned has to be integrated in order to get a final integral of an equation or system of equations, although some of them may be of an order higher than the first.

When we have to deal with equations of the second or higher orders, we find it necessary to solve a second set of auxiliary equations which have no analogue in the theory of equations of the first order. One system of this kind must be solved before passing to the first integral, another before we can get a second integral, and so on. No such system has to be solved before passing from the penultimate integral to the solution.

I have thought it desirable to prove that, in the case of equations of the first order, the system of auxiliary equations of which use is made in this paper, is completely equivalent to the system (*b*) commonly employed. But I have not succeeded in effecting a similar transformation in the case of equations of the second and higher orders, though such transformations are no doubt possible, and would be of the utmost practical value.

It may be added that the results are applicable to systems of simultaneous equations of the same or different orders.

Equations of the First Order.

1. Let z be a function of $x_1 \dots x_n$; viz., suppose there is a relation

$$\phi(z, x_1 \dots x_n) = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1);$$

then, writing p_i for $\frac{dz}{dx_i}$, we have

$$p_i = -\frac{d\phi}{dx^i} : \frac{d\phi}{dz} \quad (i = 1 \dots n).$$

These equations give, without difficulty,

$$\frac{dp_i}{dx_j} + p_j \frac{dp^i}{dz} = \frac{dp_j}{dx^i} + p_i \frac{dp_j}{dz} \quad (i, j = 1 \dots n) \dots\dots\dots\dots\dots (2),$$

which, being independent of the form of ϕ, must be satisfied whatever ϕ may be. Moreover they give

$$\frac{d\phi}{dz}(dz - p_1 dx_1 - \dots - p_n dx_n) = d\phi,$$

an exact differential.

Conversely, if (2) be satisfied, $p_1 \dots p_n$ are the derivatives with respect to $x_1 \dots x_n$ of a function z defined by some equation of the form (1). For (2) are just sufficient to ensure that

$$dz - p_1 dx_1 - \dots - p_n dx_n = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

should be integrable; viz., that the expression on the left, when multiplied by a certain integrating factor, should become an exact differential. Any integral of (3) will be an equation of the form (1) consistent with the given forms of p.

2. Suppose now that $p_1 \dots p_n$, instead of being given as functions of z, $x_1 \dots x_n$, are given by n equations of the form

$$F_i(p_1 \dots p_n, z, x_1 \dots x_n) = a_i, \ (i = 1 \dots n) \ \dots\dots\dots\dots\dots (4),$$

where a_i is a constant separated from F_i for convenience. We seek the relations between F_i which are necessary and sufficient to ensure the existence of an equation (1) consistent with them.

Let ξ represent any one of the $n+1$ quantities $z, x_1 \dots x_n$. Then we

have n equations of the form

$$\frac{dF_i}{dp_1}\cdot\frac{dp_1}{d\xi}+\ldots+\frac{dF_i}{dp_n}\cdot\frac{dp_n}{d\xi}+\frac{dF_i}{d\xi}=0.$$

The differential coefficients which form the second factors of the terms in these equations are obtained on the supposition that $p_1 \ldots p_n$ are by means of (4) expressed as functions of z, $x_1 \ldots x_n$. From these equations eliminate every $\frac{dp}{d\xi}$ except $\frac{dp_j}{d\xi}$. Thus, using Donkin's notation, we get

$$\frac{d(F_1 \ldots F_n)}{d(p_1 \ldots p_n)}\cdot\frac{dp_j}{d\xi}=-\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots \xi \ldots p_n)}.$$

We assume that

$$\frac{d(F_1 \ldots F_n)}{d(p_1 \ldots p_n)}$$

does not vanish; for, if it did, (4) would not define $p_1 \ldots p_n$. Substituting in (2), we obtain

$$\frac{d(F_1 \ldots F_i \ldots F_n)}{d(p_1 \ldots x_j \ldots p_n)}+p_j\frac{d(F_1 \ldots F_i \quad F_n)}{d(p_1 \ldots z \ldots p_n)}$$
$$=\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots x_i \ldots p_n)}+p_i\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots z \ldots p_n)}.$$

If, for convenience, we indicate $\frac{du}{dx_i}+p_i\frac{du}{dz}$ by $\frac{du}{d\bar{x}_i}$, these equations may be written

$$\frac{d(F_1 \ldots F_i \ldots F_n)}{d(p_1 \ldots \bar{x}_j \ldots p_n)}=\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots \bar{x}_i \ldots p_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Or again, if we represent the Jacobian formed from

$$\frac{d(F_1 \ldots F_n)}{d(p_1 \ldots p_n)},$$

by substituting $\bar{x}_i$ for p_j, by the symbol

$$\left|\begin{matrix} j \\ i \end{matrix}\right|$$

then (5) may be written

$$\left|\begin{matrix} i \\ j \end{matrix}\right|=\left|\begin{matrix} j \\ i \end{matrix}\right| \quad (i,j=1 \ldots n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The equations (5) or (6) are the conditions we seek; for they simply express that, when (4) are solved for $p_1 \ldots p_n$, the values of $p_1 \ldots p_n$ satisfy the equations (2) which are both necessary and sufficient to ensure the existence of an integral (1).

3. The equations (5), (6) may be put in another form. The transformation is easily effected by the use of the identity

$$\frac{d(F_1 \ldots F_r)}{d(\eta_1 \ldots \eta_r)}\cdot\frac{d(F_1 \ldots F_n)}{d(\xi_1 \ldots \xi_n)}=\sum_{j=1}^{j=n}\frac{d(F_1 \ldots F_r)}{d(\eta_1 \ldots \xi_j)}\cdot\frac{d(F_1 \ldots F_j \ldots F_n)}{d(\xi_1 \ldots \eta_r \ldots \xi_n)} \ldots(7),$$

where the j^{th} term on the right is formed from the term on the left by transposing η_r, ξ_j: and r is not greater than n. To establish (7) we notice that

$$\frac{dF}{d\eta}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} - \sum_{j=1}^{j=n}\frac{dF}{d\xi_j}\cdot\frac{d(F_1 \dots F_j \dots F_n)}{d(\xi_1 \dots \eta \dots \xi_n)}$$
$$= \frac{dF}{d\eta}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} + \sum_{j=1}^{j=n}(-)^j\frac{dF}{d\xi_j}\cdot\frac{d(F_1 \dots F_n)}{d(\eta\,.\,\xi_1 \dots \xi_{j-1}\,\xi_{j+1} \dots \xi_n)}$$
$$= \frac{d(F, F_1 \dots F_n)}{d(\eta, \xi_1 \dots \xi_n)},$$

and this vanishes when F is any one of the quantities $F_1 \dots F_n$. Now, bringing the right-hand member of (7) to the left side, and partially expanding the first factor of each term, we may write it

$$\frac{d(F_1 \dots F_{r-1})}{d(\eta_1 \dots \eta_{r-1})}\cdot\left\{\frac{dF_r}{d\eta_r}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} - \sum_{j=1}^{j=n}\frac{dF_r}{d\xi_j}\cdot\frac{d(F_1 \dots F_j \dots F_n)}{d(\xi_1 \dots \eta_r \dots \xi_n)}\right\}$$
$$-\frac{d(F_1 \dots F_{r-2}, F_r)}{d(\eta_1 \dots \eta_{r-2}, \eta_{r-1})}\cdot\left\{\frac{dF_{r-1}}{d\eta_r}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} - \sum_{j=1}^{j=n}\frac{dF_{r-1}}{d\xi_j}\cdot\frac{d(F_1 \dots F_j \dots F_n)}{d(\xi_1 \dots \eta_r \dots \xi_n}\right\}$$
$$+\&c. \qquad = 0 \dots\dots (8).$$

But the second factor of each line has just been proved to vanish provided $F_1 \dots F_r$ are any r of $F_1 \dots F_n$; this being implied in the condition that r is not greater than n, the truth of (8) or (7) is manifest.

If ξ_j should be identical with any of the $r-1$ quantities $\eta_1 \dots \eta_{r-1}$, the Jacobian

$$\frac{d(F_1 \dots F_{r-1}, F_r)}{d(\eta_1 \dots \eta_{r-1}, \xi_j)}$$

vanishes identically; and the corresponding term in (7) disappears. Hence the summation in (7) need only be extended to those quantities $\xi_1 \dots \xi_n$, which do not appear amongst the $r-1$ quantities $\eta_1 \dots \eta_{r-1}$.

4. Multiply (6) by $\dfrac{d(F_k, F_l)}{d(p_i, p_j)}$. The result may be written

$$\frac{d(F_k, F_l)}{d(p_i, p_j)}\begin{vmatrix} i \\ j \end{vmatrix} + \frac{d(F_k, F_l)}{d(p_j, p_i)}\begin{vmatrix} j \\ i \end{vmatrix} = 0.$$

Sum the results obtained by giving i, j all values from 1 to n inclusive. It is clear that the sums derived from the two terms will be identical; so that we may confine our attention to one of the terms—say the first. The summation with respect to i gives

$$\sum_{i=1}^{i=n}\frac{d(F_k\,.\,F_l)}{d(p_i\,.\,p_j)}\cdot\begin{vmatrix} i \\ j \end{vmatrix} = -\sum_{i=1}^{i=n}\frac{d(F_k\,.\,F_l)}{d(p_j\,.\,p_i)}\cdot\frac{d(F_1 \dots F_i \dots F_n)}{d(p_1 \dots \bar{x}_j \dots p_n)}$$
$$= -\frac{d(F_k, F_l)}{d(p_j, \bar{x}_j)}\cdot\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \quad \text{by (7).}$$

Then, summing with respect to j,

$$\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \cdot \sum_{j=1}^{j=n} \frac{d(F_k, F_l)}{d(\bar{x}_j, p_j)}.$$

Now, if all the equations (6) are satisfied, this sum must vanish; but the first factor cannot vanish, so that

$$\sum_{j=1}^{j=n} \cdot \frac{d(F_k, F_l)}{d(\bar{x}_j, p_j)} = 0 \ (k, l = 1 \ .. \ n) \ \dots\dots\dots\dots\dots\dots \ (9).$$

These equations will be indicated as usual by the notation

$$[F_k, F_l] = 0 \ \dots\dots\dots\dots\dots\dots\dots \ (10).$$

5. It has been proved that the system (9) is a consequence of (5). To prove the complete equivalence of these systems, it is still necessary to show that (5) is a consequence of (9). For this purpose, multiply (9) by the Jacobian formed from

$$\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)},$$

by omitting F_k, F_l $(k < l)$ and two p's—say p_κ, p_λ. Give the product a positive or negative sign according as $k+l$ is odd or even. Then sum the results for $k, l = 1 \dots n$, κ, λ being the same throughout. Call the result $\Sigma_{\kappa\lambda}$. Consider the sum derived from a single term of (9), say

$$\frac{d(F_k \cdot F_l)}{d(\bar{x}_j, p_j)}.$$

This is

$$\frac{d(F_1, F_2)}{d(\bar{x}_j, p_j)} \cdot \frac{d(F_3 \dots F_n)}{d(p_1 \dots p_n)} - \frac{d(F_1, F_3)}{d(\bar{x}_j, p_j)} \cdot \frac{d(F_2 \cdot F_4 \dots F_n)}{d(p_1 \dots p_n)} + \&c.$$
$$= \frac{d(F_1, F_2, F_3 \dots F_n)}{d(\bar{x}_j, p_j, p_1 \dots p_n)},$$

where p_κ, p_λ are supposed to be omitted in the range $p_1 \dots p_n$. Now this determinant vanishes identically if two constituents of the denominator are the same; that is, unless j is either κ or λ. Hence

$$\Sigma_{\kappa\lambda} = \frac{d(F_1 \cdot F_2 \dots F_n)}{d(\bar{x}_\kappa, p_\kappa, p_1 \dots p_n)} + \frac{d(F_1 \dots F_n)}{d(\bar{x}_\lambda, p_\lambda, p_1 \dots p_n)}$$
$$= \frac{d(F_1 \dots F_n)}{d(\bar{x}_\kappa, p_\kappa, p_1 \dots p_n)} - \frac{d(F_1 \dots F_n)}{d(p_\lambda, \bar{x}_\lambda, p_1 \dots p_n)}.$$

Now in each determinant place the second constituent of the denominator (p_κ or x_λ) under F_κ and the first under F_λ: which changes will clearly produce the same change of sign in each determinant. Thus

$$\pm \Sigma_{\kappa\lambda} = \frac{d(F_1 \dots F_\lambda \dots F_n)}{d(p_1 \dots \bar{x}_\kappa \dots p_n)} - \frac{d(F_1 \dots F_\kappa \dots F_n)}{d(p_1 \dots \bar{x}_\lambda \dots p_n)}.$$

But if (9) be true, $\Sigma_{\kappa\lambda}$ vanishes for all values of κ, λ from 1 to n inclusive, so that this last result is simply (5). Hence (9), (5)—and therefore (2)—are completely equivalent.

6. The application of these results to the integration of a system of equations of the first order is easy. It may be possible from a given system to eliminate $p_1 \dots p_n$ so as to get one or more relations between $z, x_1 \dots x_n$. These cannot be regarded as integrals; and we shall suppose the system freed from such irrelevant members by suitable transformations. The reduced system consists of not more than n equations—say the number is r. These r equations may be put in the place of $F_1 \dots F_r$ in (4), and we have to find $F_{r+1} \dots F_n$ so as to satisfy (5) or (9). This, in general, implies the solution of linear partial differential equations, the theory of which is well known. But in some cases, as Prof. Nanson has observed,* it is possible to find some of the unknown functions F without integration. We have in fact, using the system (9), certain relations

$$[F_i F_j] = 0 \quad (i, j = 1 \dots r)$$

between known functions. Now these relations may be satisfied identically, or in virtue of the given system; or they may be new equations which must be satisfied if there be any integral to the system. If in this way, or otherwise, we determine $F_{r+1} \dots F_n$, such that (10) are all satisfied, the values of $p_1 \dots p_n$ derived from the complete system will render the equation

$$dz - p_1 dx_1 - \dots\dots - p_n dx_n = 0 \quad \dots\dots\dots\dots\dots\dots \quad (3)$$

integrable, and the integral will be a solution of the given system. If we can find all the solutions of (10), then we can find all the solutions of the given system; and there is no solution to the system when there is no solution of (10).

The number of arbitrary constants in the solution is generally $n-r+1$. For each of the $n-r$ functions $F_{r+1} \dots F_n$, being determined from its differential coefficients, involves one constant, and there is a constant of integration from (3). If, however, any of $F_{r+1} \dots F_n$ are determined without integration, as above, the number of arbitrary constants is reduced.

These results agree substantially with those given by Prof. Nanson in his paper cited above, though the form, and the manner of obtaining them, are somewhat different.

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* On the Theory of the Solution of a System of Simultaneous Non-linear Partial Differential Equations of the First Order. Proc. R. S., Vol. xxiv. p. 337. [This theory does not appear to be essentially different from that previously given by Bour. See Mansion, "Théorie des équations aux dérivées partielles du premier ordre," §§ 21, 22.]

Equations of the Second Order.

7. In discussing equations of the second order, we shall start from the results obtained in the former part of this paper. It has been proved, it will be remembered, that the existence of an integral

$$\phi(z, x_1 \dots x_n) = 0 \quad \dots\dots(1)$$

requires that the first differential coefficients of z should satisfy n equations

$$F_j(p_1 \dots p_n, z, x_1 \dots x_n) = a_j \quad \dots\dots(4)$$

such that

$$\frac{d(F_1 \dots F_i \dots F_n)}{d(p_1 \dots \bar{x}_j \dots p_n)} = \frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots \bar{x}_i \dots p_n)} \quad \dots\dots(5).$$

Differentiating (4) with respect to x_i, and writing s_{ij} for $\frac{d^2z}{dx_i\,dx_j}$, we get

$$\frac{dF_j}{dp_1}s_{1i} + \dots + \frac{dF_j}{dp_n}s_{ni} + \frac{dF_j}{dz}p_i + \frac{dF_j}{dx_i} = 0.$$

From the n equations formed by putting $j = 1, 2, \dots n$ in turn, eliminate every s but s_{ij}. The result is

$$\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)}s_{ij} = -\frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots x_i \dots p_n)} - p_i\frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots z \dots p_n)}.$$

If we put

$$\left.\begin{aligned} X_{ij} &= \frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots x_i \dots p_n)} : \frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \\ Z_j &= \frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots z \dots p_n)} : \frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \end{aligned}\right\} \quad \dots\dots(11),$$

we may write this result

$$s_{ij} = -X_{ij} - p_i Z_j \quad \dots\dots(12).$$

In the same notation (5) may be written

$$X_{ji} + p_j Z_i = X_{ij} + p_i Z_j \quad \dots\dots(13),$$

so that $s_{ji} = s_{ij}$, as it should.

8. The equations

$$\frac{d(F, F_1 \dots F_n)}{d(x_i, p_1 \dots p_n)} = 0, \quad \frac{d(F, F_1 \dots F_n)}{d(z, p_1 \dots p_n)} = 0,$$

are identically satisfied when F is any one of the quantities $F_1 \dots F_n$. Expanding them, and substituting from (11), we get them in the form

$$\Delta_i F \equiv \frac{dF}{dx_i} - X_{i1}\frac{dF}{dp_1} - \dots - X_{in}\frac{dF}{dp_n} = 0,$$

$$\Delta F \equiv \frac{dF}{dz} - Z_1\frac{dF}{dp_1} - \dots - Z_n\frac{dF}{dp_n} = 0.$$

Giving i all values from 1 to n, we have a system of $n+1$ indepen-

dent equations, having $2n+1$ independent variables, and n common solutions, $F_1 \ldots F_n$. This implies the conditions

$$\left.\begin{aligned} \Delta_j X_{ik} &= \Delta_i X_{jk} \\ \Delta X_{ik} &= \Delta_i Z_k \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

for all values of i, j, k from 1 to n. Also, these conditions being satisfied, the n solutions are the integrals of the integrable system,

$$dp_i + X_{1i}\,dx_1 + \ldots + X_{ni}\,dx_n + Z_i\,dz = 0 \quad (i = 1 \ldots n) \ldots\ldots(15).$$

If in this equation we write $p_1 dx_1 + \ldots + p_n dx_n$ for dz [which is permissible, because we have assumed (5) or (13) to hold good], and reduce by means of (12), the system (15) takes the form

$$dp_i - s_{1i}\,dx_1 - \ldots - s_{ni}\,dx_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots(16).$$

9. We may also express (14) in terms of s_{ij}, &c. Observe that, in virtue of the definition of Δ_i, Δ, taking account of (12), we have

$$\Delta_i + p_i\Delta = \frac{d}{dx_i} + p_i \frac{d}{dz} + s_{i1} \frac{d}{dp_1} + \ldots + s_{in} \frac{d}{dp_n};$$

and, whatever u may be,

$$\Delta_i p_j u = p_j \Delta_i u - X_{ij} u,$$
$$\Delta\, p_j u = p_j \Delta\, u - Z_j u.$$

With the help of these relations it is easy to verify that

$$(\Delta_i + p_i\Delta)(X_{jk} + p_j Z_k) = (\Delta_j + p_j\Delta)(X_{ik} + p_i Z_k),$$

that is,

$$\left(\frac{d}{dx_i} + p_i \frac{d}{dz} + s_{i1} \frac{d}{dp_1} + \ldots + s_{in} \frac{d}{dp_n}\right) s_{jk}$$
$$= \left(\frac{d}{dx_j} + p_j \frac{d}{dz} + s_{j1} \frac{d}{dp_1} + \ldots + s_{jn} \frac{d}{dp_n}\right) s_{ik}.$$

This may be written $$\frac{ds_{jk}}{d\bar{x}_i} = \frac{ds_{ik}}{d\bar{x}_j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

if we take these differential coefficients to imply differentiation with respect to x, not only as it occurs explicitly in s, but also as involved in z and the first differential coefficients of z.

10. It has been shown that (17) are necessary consequences of the existence of an integral $\phi(z, x_1 \ldots x_r) = 0$.

We shall now investigate how far these conditions are sufficient to ensure the existence of an integral and to enable us to find it. The first step in the discussion will be to show that, when (17) are satisfied, the equations $dp_i - s_{1i}\,dx_1 - \ldots - s_{ni}\,dx_n = 0 \quad (i = 1 \ldots n) \ldots\ldots(18)$

are an integrable system. But it is obvious that in the form (18) this cannot be true, save under certain conditions; for the coefficients of the equations involve z, while the differential dz does not occur anywhere. That (18) in their present form should be an integrable system, it is necessary that each s should be free from z. But the results of Arts. 7—9 suggest the procedure to be adopted when the s's do involve z.

11. Assume $s_{ij} = -X_{ij} - p_i Z_j = -X_{ji} - p_j Z_i = s_{ji}$ .....(19),

and write
$$\Delta_i = \frac{d}{dx_i} - X_{i1}\frac{d}{dp_1} - \ldots - X_{in}\frac{d}{dp_n},$$
$$\Delta = \frac{d}{dz} - Z_1\frac{d}{dp_1} - \ldots - Z_n\frac{d}{dp_n};$$

so that (17) may be written

$$(\Delta_i + p_i\Delta)(X_{jk} + p_j Z_k) = (\Delta_j + p_j\Delta)(X_{ik} + p_i Z_k) \quad \ldots\ldots(20).$$

Here X_{ij}, Z_i, &c. are only defined by (19). If we determine Z, then every X is given, and *vice versâ*. We adopt the former course, and suppose that $Z_1 \ldots Z_n$ are determined by the equations

$$\frac{dZ_i}{d\bar{x}_j} + \frac{ds_{ij}}{dz} - Z_1\frac{ds_{ij}}{dp_1} - \ldots - Z_n\frac{ds_{ij}}{dp_n} - Z_i Z_j = 0 \quad \ldots\ldots(21),$$

or, as it may be written,

$$(\Delta_j + p_j\Delta)Z_i = \Delta(X_{ji} + p_j Z_i) + Z_i Z_j \quad \ldots\ldots(22).$$

Of these equations we shall have to say a few words hereafter (Art. 15). For the present we suppose that $Z_1 \ldots Z_n$ are determined by their help; (21) involving, besides $Z_1 \ldots Z_n$, only quantities given.

Now (22) gives on reduction

$$\Delta_j Z_i = \Delta X_{ji} \quad \ldots\ldots(23);$$

and in virtue of this (20) reduces to

$$\Delta_i X_{jk} = \Delta_j X_{ik} \quad \ldots\ldots(24),$$

as may easily be seen by expanding it so that the Δ's act only on the X's and Z's; and then reducing it by the help of (23) and (19).

The relations (23), (24) show that, whatever u may be,

$$\left.\begin{aligned}\Delta_i\Delta_j u &= \Delta_j\Delta_i u\\ \Delta_i\Delta\, u &= \Delta\,\Delta_i u\end{aligned}\right\} \quad \ldots\ldots(25).$$

For, taking the first of these, consider the coefficients of $\frac{du}{dp_k}$. They are $-\Delta_i X_{jk}$ and $-\Delta_j X_{ik}$ respectively, and by (24) are equal. Again, the coefficient of $\frac{d^2u}{dp_k\,dp_l}$ on each side is $X_{ik}X_{jl} + X_{jk}X_{il}$. Hence the two sides are equal term to term. Similarly for the second of (25).

12. Form the $n+1$ equations,

$$\Delta F = 0, \quad \Delta_i F = 0 \quad (i = 1 \dots n) \quad \dots\dots\dots\dots (26),$$

having $2n+1$ independent variables. Since (25) are true, these equations have the maximum number (n) of common solutions. If we write for Δ, Δ_i their forms, given at the beginning of Art. 11, it is clear that the n solutions of (26) are to be found by integrating the system of ordinary differential equations,

$$dp_i + X_{1i}\,dx_1 + \dots + X_{ni}\,dx_n + Z_i\,dz = 0 \quad \dots\dots\dots\dots (27).$$

These n equations form an integrable system. We do not say that any equation of the system is an exact differential, or is even integrable; but that from the system we can form, by suitable combinations, n independent exact differential equations. For, if not, (26) would have less than n common solutions; that is, (25) would not be true.

This system (27) it is which must take the place of (18); and it will be noticed that the difficulty which was found in employing (18) is now entirely removed.

13. Let the n common solutions of (26), or the n integrals of (27), be indicated by

$$F_i\,(p_1 \dots p_n,\ z,\ x_1 \dots x_n) = a_i \quad (i = 1 \dots n) \quad \dots\dots\dots (28).$$

The general solution of (26) is

$$\phi\,(F_1, \dots F_n) = 0,$$

where ϕ is arbitrary. If from this by differentiation we eliminate ϕ, we reproduce the equations (26); and comparison of coefficients gives

at once
$$\left.\begin{aligned} X_{ji} &= \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots x_j \dots p_n)} : \frac{d\,(F_1 \dots F_n)}{d\,(p_1 \dots p_n)} \\ Z_i &= \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots z \dots p_n)} : \frac{d\,(F_1 \dots F_n)}{d\,(p_1 \dots p_n)} \end{aligned}\right\} \quad \dots\dots\dots (29).$$

Hence (19) becomes

$$\frac{d\,(F_1 \dots F_j \dots F_n)}{d\,(p_1 \dots x_i \dots p_n)} + p_i \frac{d\,(F_1 \dots F_j \dots F_n)}{d\,(p_1 \dots z \dots p_n)} = \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots x_j \dots p_n)} + p_j \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots z \dots p_n)}.$$

But these are precisely the conditions which are necessary and sufficient to ensure that the values of $p_1 \dots p_n$ derived from (28) are such as to render
$$dz - p_1 dx_1 - \dots - p_n dx_n = 0$$
integrable.

14. The results obtained may be summed up thus :—Let $s_{12} \dots s_{ij} \dots s_{nn}$ be given as functions of $x_1 \dots x_n$, z, $p_1 \dots p_n$, such that

$$\frac{ds_{jk}}{d\bar{x}_i} = \frac{ds_{ik}}{d\bar{x}_j} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(17).$$

Determine $Z_1 \dots Z_n$ from the n^2 equations

$$\frac{dZ_i}{d\bar{x}_j} + \frac{ds_{ij}}{dz} - Z_1 \frac{ds_{ij}}{dp_1} - \dots - Z_n \frac{ds_{ij}}{dp_n} - Z_i Z_j = 0 \quad \dots\dots\dots\dots(21),$$

and X_{ji} from the equations

$$X_{ji} = p_j Z_i - s_{ji}.$$

Then the system

$$dp_i + X_{1i} dx_1 + \dots + X_{ni} dx_n + Z_i dz = 0 \quad \dots\dots\dots\dots\dots(27)$$

is integrable; and the values of $p_1 \dots p_n$ determined from the n integrals of this system are such as to make

$$dz - p_1 dx_1 - \dots - p_n dx_n = 0 \quad \dots\dots\dots\dots\dots\dots\dots(3)$$

integrable. Any integral of this equation is a solution of the given system.

15. We revert for a moment to the consideration of the system (21) by means of which $Z_1 \dots Z_n$ are determined. This system, consisting of n^2 equations, involves n dependent variables, and $2n+1$ independent variables. Now it may very well be that the system does not always admit of being satisfied by any values of $Z_1 \dots Z_n$; in other words, these equations may not be consistent unless certain conditions are fulfilled which are not implied in (17). It will then be well to point out that such conditions, should they exist, are necessary conditions for the existence of a solution of the proposed system of equations, and have not been introduced by any arbitrary procedure peculiar to the method of this paper. The results of Art. 13 show that the X, Z of Arts. 11—14 are actually identical with the X, Z of Arts. 7—9; and, therefore, that the Δ, Δ_i of these Articles are also the same. Now the equation (21) written in the form

$$(\Delta_j + p_j \Delta) Z_i = \Delta (X_{ji} + p_j Z_i) + Z_i Z_j \quad \dots\dots\dots\dots(22),$$

or

$$\Delta_j Z + p_j \Delta Z_i = \Delta X_{ji} + p_j \Delta Z_i,$$

is simply one of the equations (14). But (14) have been proved to be a consequence of the existence of an integral. Hence the equations (21) must be satisfied if there be any integral; and if there be no solutions of (21), neither can there any integral corresponding to the given values of $s_{11} \dots s_{ij} \dots s_{nn}$.

It may be desirable to show how to obtain an equation for each of the quantities $Z_1, \dots Z_n$ freed from all the others. This may be done

by writing (21) in the form

$$\frac{dZ_i}{d\bar{x}_j}+\frac{ds_{ij}}{dz}-Z_i\frac{ds_{ij}}{dp_i}-Z_1\frac{ds_{ij}}{dp_1}-\ldots-Z_j\left(\frac{ds_{ij}}{dp_j}+Z_i\right)-\ldots-Z_n\frac{ds_{ij}}{dp_n}=0.$$

In this give j all values from 1 to n inclusive; and from the n equations thus formed eliminate every Z but Z_i. The result will be a partial differential equation of the first order for Z_i which is linear in the differential coefficients of Z_i.

It will be observed that in (21) all the terms save the first are symmetrical with respect to i, j. Hence we get

$$\frac{dZ_i}{d\bar{x}_j}=\frac{dZ_j}{d\bar{x}_i},$$

equations which are equivalent to

$$Z_i=\frac{dZ}{d\bar{w}_i}.$$

Hence it appears that the problem of finding $Z_1, \ldots Z_n$ is reducible to the problem of finding one quantity only. In fact, I have not found any advantage resulting from this remark; but it may possibly be of value in considering the theory of equations of higher orders.

16. In the preceding paragraphs it has been supposed that $s_{11} \ldots s_{ij} \ldots s_{nn}$ have been given in terms of $p_1 \ldots p_n, z, x_1 \ldots x_n$. Let us now take the more general case in which these functions are given by $\frac{n(n+1)}{2}$ equations of the form

$$F_{ij}(s_{11} \ldots s_{ij} \ldots s_{nn},\ p_1 \ldots p_n, z, x_1 \ldots x_n) = b_{ij} \quad \ldots\ldots\ldots\ldots (30).$$

It is required to find the relations between F_{ij} which are equivalent to the conditions (17). This necessitates a transformation exactly like that of Art. 2, and the result is

$$\frac{d(F_{11} \ldots F_{jk} \ldots F_{nn})}{d(s_{11} \ldots \bar{x}_i \ldots s_{nn})}=\frac{d(F_{11} \ldots F_{ik} \ldots F_{nn})}{d(s_{11} \ldots \bar{x}_j \ldots s_{nn})} \quad \ldots\ldots\ldots\ldots (31),$$

the differentiations with respect to x_i, x_j being made on the supposition that $z, p_1 \ldots p_n$ are functions of x_i, x_j.

These equations may be indicated by an abbreviation similar to that used in (6); viz.

$$\left|\begin{matrix} jk \\ i \end{matrix}\right| = \left|\begin{matrix} ik \\ j \end{matrix}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32),$$

where the first member, for example, indicates the result of substituting $\bar{x}_i$ for s_{jk} in the Jacobian

$$\frac{d(F_{11} \ldots F_{jk} \ldots F_{nn})}{d(s_{11} \ldots s_{jk} \ldots s_{nn})}.$$

17. These results may be applied to the solution of a system of equations of the second order in a manner similar to that employed in Art. 6. The system may be such that it is possible to form from it equations between $z, x_1 \dots x_n$; but as these cannot be regarded as integrals, they will be neglected, and we shall suppose the system freed from them. Again, it may be possible to get relations between $p_1 \dots p_n, z, x_1 \dots x_n$; or such equations may explicitly occur as members of the system. The treatment of such cases will be noted in the next Article; and for the present we assume that the system comprises r $\left(\text{not more than } \frac{n(n+1)}{2}\right)$ equations from which the differential coefficients of the second order cannot be eliminated.

We have then to find $\frac{n(n+1)}{2} - r$ functions F, either by artifices like those mentioned in Art. 6, or by actual integration of (31) in some one of its forms. If this can be accomplished, nothing remains but to solve for $s_{11} \dots s_{ij} \dots s_{nn}$ from the completed system, and proceed as in Art. 14 to the final integral.

18. If any equations of the first order be included in the system; the procedure is not essentially changed. Each such equation should be differentiated with respect to $x_1, \dots x_n$ in turn; thus furnishing n equations of the second order. Of course some or all of these may be included in the given equations of the second order. The system thus formed is to be treated as in Art. 17, but the arbitrary constants of the first integrals must be determined so as to agree with the given equations of the first order.

19. As to the number of arbitrary constants. Observe that $F_{11} \dots F_{nn}$ each involve one arbitrary constant (b) because each is determined from its differential coefficients. Now suppose there are r_1 equations of the first order, and r_2 of the second order (taking account of those derived from the r_1 equations of the first order). We shall then have

$$\frac{n(n+1)}{2} - r_2$$

constants from the system (30). We shall have

$$n - r_1$$

constants, of integration, from (27); and one from the integration of (3). Thus altogether there will be

$$\frac{n(n+1)}{2} + n + 1 - r_2 - r_1$$

arbitrary constants in the solution.

I have used the phrase "arbitrary constants" in the above passage

and elsewhere; but without qualification this would be wrong. The quantities indicated are not constants at all, though they include constants as a particular case. They really are functions of the variables defined by certain differential equations, which are well known when the equations in whose solution they occur are of the first order; but have not been determined, so far as I know, for equations of higher orders save in special cases.

Equations of the Third and Higher Orders.

20. It would be possible, and perhaps desirable, to discuss the equation of the third order in a manner similar to that in which equations of the second order were dealt with. But I purpose changing the method so as to make it applicable at once to equations of higher orders, and the remarks will be brief.

Evidently the equation

$$\frac{d^4 z}{dx_i\, dx_j\, dx_k\, dx_l} = \frac{d^4 z}{dx_j\, dx_i\, dx_k\, dx_l} \quad \text{.....(33)}$$

is identically true. Let us see what it becomes in terms of the differential coefficients of the third order, supposed to be given in terms of s, p, z, x. Write these differential coefficients a_{ijk}, &c., the subscripts indicating the independent variables which are concerned. Then (33) may be written

$$\frac{da_{jkl}}{dx_i} = \frac{da_{ikl}}{dx_j} \quad \text{.....(34),}$$

where x, $p_1 \dots p_n$, $s_{11} \dots s_{nn}$ are all regarded as functions of x_i, x_j. If a should be given by $\frac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3}$ equations of the form

$$F_{ijk}\,(\dots a \dots s \dots p \dots z \dots x \dots) = c_{ijk} \quad \text{.....(35);}$$

then F_{ijk} must satisfy the relations

$$\left|\begin{matrix} jkl \\ i \end{matrix}\right| = \left|\begin{matrix} ikl \\ j \end{matrix}\right| \quad \text{.....(36),}$$

where the first member indicates the result of substituting x_i for a_{jkl} in

$$\frac{d\,(F_{111} \dots F_{jkl} \dots F_{nnn})}{d\,(a_{111} \dots a_{jkl} \dots a_{nnn})},$$

and, as before, every s, p, z is regarded as a function of x_i.

21. Suppose that we know $F_{111} \dots F_{nnn}$. They may be given; or some of them may be given, and the rest discovered by the integration of some system equivalent to (36). Without distinguishing between these cases, suppose F known, so that we can express a_{jkl}, &c. in terms of s, p, z, x. Form the equations

$$ds_{ij} - a_{ij1}\, dx_1 - \dots - a_{ijn}\, dx_n = 0 \quad \text{.....(37).}$$

This system must be equivalent to an integrable system if there be any solution consistent with the values of F_{ijk}. To find this equivalent system, write

$$-a_{ijk} = X_{ijk} + p_i Z_{jk} + p_j Z_{ki} + p_k Z_{ij} + s_{ij} P_k + \dots \quad \dots\dots\dots(38),$$

leaving X, Z, P indeterminate. Substitute these values in (37), and reduce by means of

$$p_1 dx_1 + \dots + p_n dx_n = dz,$$
$$s_{1i} dx_1 + \dots + s_{ni} dx_n = dp_i,$$
$$\&c.,$$

and we obtain a system of equations of the form

$$ds_{ij} + \dots + X_{ijk}\, dx_k + \dots + Z_{ij}\, dz + P_i dp_j + P_j\, dp_i = 0 \quad \dots\dots(39).$$

Let us now write down the conditions that this should be an integrable system, and regard the equations thus obtained, combined with (38) and with

$$a_{ijk} = a_{jik} = \&c. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(39),$$

as equations for the determination of X, Z, P. If values can be obtained, then we can find a system of first integrals; and these in virtue of (39) are sufficient to determine s_{ij} so as to yield a system of second integrals. But the determination of this second system requires a transformation analogous to that of the present article.

If, on the other hand, the equations by which X, Z, P are to be found should be inconsistent, we may be sure that there is no integral of the given equations; because all the equations for X, Z, P are necessary consequences of the existence of an integral

$$\phi\,(z, x_1 \dots x_n).$$

22. As was indicated above, these remarks are applicable at once to equations of the m^{th} order.

It may be permissible here to notice that, in the solution of a system of equations of the m^{th} order, the number of arbitrary constants is

$$\frac{\lfloor m+n-1}{\lfloor m \; \lfloor n-1} + \frac{\lfloor m+n-2}{\lfloor m-1 \; \lfloor n-1} + \&c. + \frac{(n+1)\,n}{1.2} + n + 1 - \sum_{i=1}^{i=m} r_i,$$

where r_i is the number of independent equations of the i^{th} order, counting not only those actually given, but also such as are derived by differentiation from given equations of lower orders.

On the Conditions for Steady Motion of a Fluid.

By Prof. Horace Lamb, M.A. (Adelaide).

[*Read February 14th*, 1878.]

1. In order that a given state of motion of a fluid should be a possible state of *steady* motion, it is necessary that certain conditions should be satisfied. These I proceed to investigate.

In any case of steady motion the stream-lines are the actual paths of the particles. Now let us consider any vortex-line and the system of stream-lines passing through the various points of it. These constitute a surface; and, since the vortex-lines move with the fluid, it is plain that an infinite number of vortex-lines can be drawn on this surface. Hence it must be possible to draw in the fluid a system of surfaces each of which is covered by a network of vortex-lines and stream-lines.

Next, let us take any point P on such a surface, and draw an infinitely short normal PP' ($=\nu$, say) to meet a consecutive surface of the system in P'. Along the stream-line through P let us take a series of points P_1, P_2, &c., at intervals corresponding to the same infinitely short time dt. The vortex-lines through these points, with the corresponding lines on the consecutive surface, mark out the boundaries of a series of vortex-filaments which may (in virtue of the steady motion) be looked upon as consecutive forms assumed by the same portion of fluid. The breadth, at any point, of such a filament is $q \sin\theta\, dt$, where q is the fluid velocity, and θ the angle between the stream- and vortex-lines at that point, and its depth is ν. Now Helmholtz and Thomson have shown that the product of the angular velocity into the cross-section has the same value at all points of a vortex-filament, and that this value does not alter with the time. Hence the product $q\omega \sin\theta \,.\, \nu$ (where ω is the angular velocity of the fluid) does not vary as we pass from P along either the stream-line or the vortex-line, that is, it is constant over the surface above defined.

It is assumed, in the above, that the external impressed forces have a single-valued potential, and that the density is either constant or a function of the pressure. It is, of course, necessary, in addition, that the equation of continuity should be satisfied, and that the boundary conditions should be constant with the steady motion.

2. The conditions of the preceding article are satisfied identically in the case of irrotational motion.

In the motion of a liquid in two dimensions, the product $q\nu$ is constant along any stream-line, and $\theta = \frac{\pi}{2}$; the said conditions then

reduce to this, that ω must be constant along each stream-line. The analytical expression of this is, if ψ be the stream-line function,

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = F(\psi) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

an arbitrary function of ψ.

When the motion of a liquid takes place in a series of planes passing through a certain axis, and is the same in each such plane, we have $2\pi r\nu . q$ constant along a stream-line, where r denotes the distance of any point from the axis. Also $\theta = \frac{\pi}{2}$, as before; the condition for steady motion then is

$$\frac{\omega}{r} = \text{constant along any stream-line.}$$

If ψ be the stream-line function, we have

$$\omega = -\frac{1}{r}\left(\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dz^2}\right),$$

the axis of symmetry being taken as axis of z, so that the condition becomes

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dz^2} = r^2 F(\psi) \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Conditions (1) and (2) were given by Stokes, *Camb. Phil. Trans.*, t. vii., 1842.

3. The equations of Art. 1 may be deduced from the ordinary hydrodynamical equations. In the case of steady motion, these may be written

$$u\frac{du}{dx} + v\frac{dv}{dx} + w\frac{dw}{dx} - 2(v\zeta - w\eta) = -\frac{dV}{dx} - \frac{1}{\rho}\frac{dp}{dx}, \text{ \&c. \&c.,}$$

where ξ, η, ζ denote the component angular velocities, viz.,

$$\xi = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \text{ \&c. \&c.}$$

Hence, if we write

$$P = \int \frac{dp}{\rho} + V + \tfrac{1}{2}q^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

we have

$$\frac{dP}{dx} = 2(v\zeta - w\eta),$$

$$\frac{dP}{dy} = 2(w\xi - u\zeta),$$

$$\frac{dP}{dz} = 2(u\eta - v\xi).$$

These equations, when interpreted, give the results of Art. 1, and shew moreover that the quantity P defined by (3) is constant over each member of the system of surfaces there defined; an extension of the well-known theorem that P is constant along a stream-line.

March 14th, 1878.

Lord RAYLEIGH, F.R.S., President, in the Chair.

Mr. Artemas Martin, M.A., Erie, Pennsylvania, was proposed for election.

Mr. Tucker communicated a paper by Prof. J. Clerk Maxwell, F.R.S., "On the Electrical Capacity of a long narrow Cylinder, and of a Disk of sensible Thickness."

Prof. Cayley, Messrs. Glaisher, Roberts, and the Chairman made short communications to the Society.

The following presents were received:—

"Greek Geometry, from Thales to Euclid," by G. Johnston Allman, LL.D.: from the Author (extracted from "Hermathena," Vol. iii., No. V.).

"Sixth General Report of the Association for the Improvement of Geometrical Teaching," Jan. 1878.

Reprint of Mathematics from the "Educational Times," Vol. xxviii.: from the Publishers.

Proceedings of Royal Society, Vol. xxvii., No. 185.

"Monatsbericht," Dec. 1877.

"Bulletin des Sciences Mathématiques et Physiques," 2^{me} Série, Tome i., Nov. 1877.

"Crelle," 84 Band, 4es Heft.

"Atti della R. Accademia dei Lincei," Serie 3^{za}; "Transunti," Vol. ii., Fasc. 3°, Febb. 1878.

"Tidsskrift for Mathematik udgivet af H. G. Zeuthen" (Fjerde Række, Første Aargang, Første, andet, Tredie, Fjerde, Femte, and Sjette Hefte); Kjøbenhavn, 1877.

"Boletin de la Institucion libre de Enseñanza," Nos. (2, 4, 5) bis, 6—12, 14—25; Madrid, 1877–78.

"Beweis eines Lehrsatzes betreffend die Integration algebraischer Differential-ausdrücke beziehungsweise algebraischer Differentialgleichungen unter geschlossener Form," von Dr. Julius Petersen (Nachrichten von der königl. Gesellschaft der Wissenschaften und der G. A. Universität zu Göttingen, 20 Febr., No. 3, 1878), Kopenhagen: from the Author.

On the Electrical Capacity of a long narrow Cylinder, and of a Disk of sensible Thickness. By Prof. Clerk Maxwell, F.R.S.

[*Read March 14th*, 1878.]

The distribution of electricity in equilibrium on a straight line without breadth is a uniform one. We may expect, therefore, that the distribution on a cylinder will approximate to uniformity as the radius of the cylinder diminishes.

Let $2l$ be the length of the cylinder, and b its radius.

Let x be measured along the axis from the middle point of the axis, and let y be the distance of any point from the axis.

Let λ be the linear density on the curved surface of the cylinder; that is, let λdx be the charge on the annular element dx.

Let σ be the surface-density on the flat ends.

Then, at a point on the axis for which $\xi = x$, the potential (ψ) is

$$\psi = \int_{-l}^{+l} \lambda \left[(\xi-x)^2+b^2\right]^{-\frac{1}{2}} dx + 2\pi \int_0^b \sigma\eta \left[(l-\xi)^2+y^2\right]^{-\frac{1}{2}} dy + 2\pi \int_0^b \sigma\eta \left[(l+\xi)^2+y^2\right]^{-\frac{1}{2}} dy,$$

the first integral being extended over the curved surface, and the other two over the positive and negative flat ends respectively.

If the electricity is in equilibrium in the cylinder, ψ must be constant for all points within the cylinder, and therefore for all points on the axis.

Also, by Art. 144 of "Electricity and Magnetism," if ψ is constant for all points on the axis, it is constant for all points within the surface of the cylinder.

If we suppose λ and σ constant,

$$\psi = \lambda \log \frac{(f_1+l-\xi)(f_2+l+\xi)}{b^2} + 2\pi\sigma (f_1+f_2-2l),$$

where f_1 and f_2 are the distances of the point (ξ) on the axis from the positive and negative edges of the curved surface.

At the middle of the axis,

$$\xi = 0, \quad \text{and} \quad f_1 = f_2 = \sqrt{l^2+b^2},$$

$$\psi_{(0)} = 2\lambda \log \frac{f+l}{b} + 4\pi\sigma (f-l).$$

At either end of the axis, $\xi = l$, $f_1 = b$, $f_2 = 2l$, nearly,

$$\psi_{(l)} = \lambda \log \frac{4l}{b} + 2\pi\sigma b.$$

Just within the cylinder, when ξ is just less than l,

$$\frac{d\psi}{dx_{(l)}} = -\lambda\left(\frac{1}{b}-\frac{1}{f}\right)+2\pi\sigma = 0 \quad \text{by hypothesis.}$$

Hence $$2\pi b\sigma = \lambda,$$

or the density on the end must be equal to that on the curved surface.

The whole charge is therefore $E = 2\pi b\sigma\,(2l+b)$.

The greatest potential is $$\psi_{(0)} = 2\pi b\sigma\left(2\log\frac{2l}{b}+\frac{b}{l}\right).$$

The smallest potential is that at the curved edge, and is approximately

$$\psi_{(e)} = 2\pi b\sigma\left(\log\frac{4l}{b}+\frac{2}{\pi}\right);$$

and the capacity must lie between

$$\frac{E}{\psi_{(0)}} = \frac{2l+b}{2\log\frac{2l}{b}+\frac{b}{l}} \quad \text{and} \quad \frac{E}{\psi_{(e)}} = \frac{2l+b}{\log\frac{4l}{b}+\frac{2}{\pi}}.$$

These are the limits between which Cavendish shows that the capacity must lie. When the cylinder is very narrow, the upper limit is nearly double the lower, so that we cannot obtain in this way any approximation to the true value.

To obtain an approximation, we may make use of the following method, in which we neglect the effect of the flat ends, and consider the cylinder as a hollow tube:—

Let Q be the potential energy of any arbitrary distribution of electricity on the cylinder,

$$Q = \tfrac{1}{2}\int_{-l}^{+l}\lambda\psi d\xi.$$

The charge is $$E = \int_{-l}^{+l}\lambda d\xi.$$

Let us now suppose this charge to distribute itself so as to pass into the state of equilibrium; then the potential will become uniform, and equal, say, to ψ_0, and $$Q_0 = \tfrac{1}{2}\psi_0 E.$$

If K is the capacity of the conductor,

$$E = K\psi_0, \quad \text{and} \quad K = \frac{1}{2}\frac{E^2}{Q_0}.$$

Since Q, the potential energy due to any arbitrary distribution of the charge, may be greater, but cannot be less, than Q_0, the energy of

the same charge in equilibrium, the capacity may be greater, but cannot be less, than

$$\frac{1}{2}\frac{E^2}{Q} \quad \text{or} \quad \frac{\left[\int_{-l}^{+l}\lambda d\xi\right]^2}{\int_{-l}^{+l}\lambda\psi d\xi}.$$

This inferior limit of the capacity is greater than that derived from the maximum value of the potential, and, as we shall see, often gives a very close approximation to the truth. Thus, if we suppose, in the case of the cylinder, λ to be uniform,

$$E = 2\lambda l, \qquad Q = 2\lambda^2 l\left(\log\frac{f+2l}{b} - \frac{f-b}{2l}\right),$$

where $f^2 = 4l^2 + b^2$. For a long narrow cylinder,

$$K_0 > \frac{l}{\log\frac{4l}{b} - 1}.$$

To obtain a closer approximation, let us suppose the distribution to be of any form, and to be expressed in the form of a series of harmonics.

The potential due to any such distribution at a given point may be expressed in terms of spherical harmonics of the second kind. See Ferrers' "Spherical Harmonics," Chap. v.

If we write $\qquad \alpha = \frac{1}{2}\frac{r_2+r_1}{l}, \quad \beta = \frac{1}{2}\frac{r_2-r_1}{l},$

where r_1 and r_2 are the distances of a given point from the ends of the line, and if the linear density is expressed by

$$\lambda = \Sigma A_i P_i\left(\frac{\xi}{l}\right),$$

where P_i is the zonal harmonic of degree i, then the potential at the given point (α, β) is $\qquad \psi = \Sigma A_i\, Q_i(\alpha)\, P_i(\beta),$

where Q_i is the zonal harmonic of the second kind, and is of the form

$$Q_i(\alpha) = P_i(\alpha)\log\frac{\alpha+1}{\alpha-1} + R_i(\alpha),$$

where $R_i(\alpha)$ is a rational function of α of $(i-1)$ degrees, and is such that $Q_i(\alpha)$ vanishes when α is infinite, thus:

$$Q_0(a) = \log \frac{a+1}{a-1},$$

$$Q_1(a) = a \log \frac{a+1}{a-1} - 2,$$

$$Q_2(a) = \left(\frac{3}{2}a^2 - \frac{1}{2}\right) \log \frac{a+1}{a-1} - 3a,$$

$$Q_3(a) = \left(\frac{5}{2}a^3 - \frac{3}{2}a\right) \log \frac{a+1}{a-1} - 5a^2 + \frac{4}{3},$$

$$Q_4(a) = \left(\frac{5.7}{8}a^4 - \frac{3.5}{4}a^2 + \frac{3}{8}\right) \log \frac{a+1}{a-1} - \frac{5.7}{4}a^3 + \frac{5.11}{12}a.$$

At a point at a very small distance b from the line, if we write

$$L = \log \frac{r_1+l-\xi}{b} + \log \frac{r_2+l+\xi}{b},$$

the potential due to the distribution whose linear density is

$$\lambda_i = A_i P_i\left(\frac{x}{l}\right)$$

is approximately

$$\psi_i = A_i P_i\left(\frac{\xi}{l}\right)\left[L - 2\left(i - \frac{i(i-1)}{2.2} + \frac{i(i-1)(i-2)}{2.3.3} - \frac{i(i-1)(i-2)(i-3)}{2.3.4.4} + \&c.\right)\right];$$

thus, if

$$\lambda_0 = A_0,$$

$$\lambda_1 = A_1 \frac{x}{l},$$

$$\lambda_2 = A_2\left(\frac{3}{2}\frac{x^2}{l^2} - \frac{1}{2}\right),$$

$$\lambda_3 = A_3\left(\frac{5}{2}\frac{x^3}{l^3} - \frac{3}{2}\frac{x}{l}\right),$$

$$\lambda_4 = A_4\left(\frac{5.7}{8}\frac{x^4}{l^4} - \frac{3.5}{4}\frac{x^2}{l^2} + \frac{3}{8}\right);$$

then approximately

$$\psi_0 = A_0 L,$$

$$\psi_1 = A_1 \frac{\xi}{l}(L-2),$$

$$\psi_2 = A_2\left(\frac{3}{2}\frac{\xi^2}{l^2} - \frac{1}{2}\right)(L-3),$$

$$\psi_3 = A_3\left(\frac{5}{2}\frac{\xi^3}{l^3} - \frac{3}{2}\frac{\xi}{l}\right)\left(L - \frac{11}{3}\right),$$

$$\psi_4 = A_4\left(\frac{5.7}{8}\frac{\xi^4}{l^4} - \frac{3.5}{4}\frac{\xi^2}{l^2} + \frac{3}{8}\right)\left(L - \frac{25}{6}\right).$$

If we write $\mathfrak{L}$ for $\log \frac{(4l^2+b^2)^{\frac{1}{2}}+2l}{b}$, or approximately $\mathfrak{L} = \log \frac{4l}{b}$, we find, to the same degree of approximation,

$$\int \lambda_0 \psi_0 \, d\xi = 4A_0^2 \, l \, (\mathfrak{L}-1),$$

$$\int \lambda_0 \psi_2 \, d\xi = \int \lambda_2 \psi_0 \, d\xi = -\tfrac{2}{3} A_0 A_2 \, l,$$

$$\int \lambda_2 \psi_2 \, d\xi = 2A_2^2 \, l \left(\tfrac{2}{3}\mathfrak{L} - \frac{101}{75} \right),$$

$$\int \lambda_0 \psi_4 \, d\xi = -\tfrac{1}{5} A_0 A_4 \, l,$$

$$\int \lambda_2 \psi_4 \, d\xi = -\tfrac{2}{7} A_2 A_4 \, l,$$

$$\int \lambda_4 \psi_4 \, d\xi = 2A_4^2 \, l \left(\tfrac{2}{9}\mathfrak{L} - \frac{6989}{5670} \right).$$

Determining A_2 so as to make $\int (\lambda_0+\lambda_2)(\psi_0+\psi_2)\, d\xi$ a minimum, and remembering that $E = 2lA_0$, we find

$$A_2 = A_0 \frac{5}{6} \cdot \frac{1}{\mathfrak{L} - \frac{101}{30}},$$

and we obtain as a second approximation

$$K_2 > \frac{l}{\mathfrak{L} - 1 - \frac{5}{36} \frac{1}{\mathfrak{L} - \frac{101}{30}}}.$$

Unless the length of the cylinder considerably exceeds 7·245 times its diameter, this approximation is of little use, for this ratio makes A_2 infinite. It shows, however, that when the ratio of the length to the diameter increases without limit, the electric density becomes more nearly uniform, and the expression for K_0 approximates to the true capacity.

We may go on to a third approximation by determining A_2 and A_4 so that $\int (\lambda_0+\lambda_2+\lambda_4)(\psi_0+\psi_2+\psi_4)\, d\xi$ shall be a minimum; whence

$$A_2 = A_0 \frac{5}{6} \frac{\mathfrak{L} - \frac{3373}{630}}{\left(\mathfrak{L} - \frac{101}{30}\right)\left(\mathfrak{L} - \frac{6989}{1260}\right) - \frac{45}{196}},$$

$$A_4 = A_0 \frac{9}{20} \frac{\mathfrak{L} - \frac{457}{210}}{\left(\mathfrak{L} - \frac{101}{30}\right)\left(\mathfrak{L} - \frac{6989}{1260}\right) - \frac{45}{196}},$$

and

$$K_4 > \frac{l}{\mathfrak{L} - 1 - \frac{5}{36}\frac{1}{\mathfrak{L} - \frac{101}{30}} - \frac{9}{400}\frac{\left(\mathfrak{L} - \frac{457}{210}\right)^2}{\left(\mathfrak{L} - \frac{101}{30}\right)\left[\left(\mathfrak{L} - \frac{101}{30}\right)\left(\mathfrak{L} - \frac{6989}{1260}\right) - \frac{45}{196}\right]}}.$$

When $\mathfrak{L}$ is very great, the distribution of electricity is expressed by the equation

$$\lambda = A_0 \left[1 + \frac{1}{\mathfrak{L}} \frac{7}{32} \left\{ 9 \frac{x^4}{l^4} - 2 \frac{x^2}{l^2} - \frac{17}{15} \right\}\right],$$

which shows that, as the ratio of the length to the diameter increases, the density becomes more nearly uniform, and the deviation from uniformity becomes more confined to the parts near the ends of the cylinder.

To indicate the character of the approximation, I have calculated $\mathfrak{L}$ and the three terms of the denominator of K_4 for different values of the ratio of l to b. When this ratio is less than 100, the third term is unavailable.

$\frac{l}{b}$	$\mathfrak{L}$	1st term.	2nd term.	3rd term.
10	3·68888	2·68888	— 0·43151	
20	4·38203	3·38203	— 0·13680	
30	4·78749	3·78749	— 0·09775	
50	5·29832	4·29832	— 0·07191	
100	5·99146	4·99146	— 0·05291	— 0·13566
1000	8·29405	7·29405	— 0·02818	— 0·00892.

Examples of the application of the method to the calculation of the capacities of a cylinder in presence of a plane conducting surface, and in presence of another equal cylinder, will be given in the notes to the forthcoming edition of Cavendish's "Electrical Researches," as illustrations of measurements made by Cavendish in 1771.

Electric Capacity of a Disk of sensible Thickness.

We may apply the same method to determine the capacity of a disk of radius a and thickness b, b being very small compared with a.

We may begin by supposing that the density on the flat surfaces is the same as when the disk is infinitely thin.

Let α and β be the elliptic coordinates of a given point with respect

to the lower disk, or in other words let the greatest and least distances of the point from the edge of the disk be $a(\alpha+\beta)$ and $a(\alpha-\beta)$.

The distance of the given point from the axis is

$$r = a\alpha\beta \quad \ldots\ldots (1),$$

and its distance from the plane of the lower disk is

$$z = a(a^2-1)^{\frac{1}{2}}(1-\beta^2)^{\frac{1}{2}} \quad \ldots\ldots (2).$$

If we write

$$a^2p^2 = a^2-r^2 \quad \ldots\ldots (3),$$

then, if A_2 is the charge of the upper disk, distributed as when undisturbed by the lower disk, the density at any point is

$$\sigma = \frac{A_2}{2\pi a^2 p} \quad \ldots\ldots (4).$$

If A_1 is the charge of the lower disk, also undisturbed, the potential at the given point due to it is

$$\psi = A_1 a^{-1} \operatorname{cosec}^{-1} a \quad \ldots\ldots (5),$$

or, if we write

$$a^2 = \gamma^2+1 \quad \ldots\ldots (6),$$

$$\psi = A_1 a^{-1}\left(\frac{\pi}{2} - \tan^{-1}\gamma\right) \quad \ldots\ldots (7).$$

We have next to find the relation between p and γ when the given point is in the upper disk, and therefore $z = b$.

Equation (2) becomes

$$b^2 = a^2\gamma^2(1-\beta^2) \quad \ldots\ldots (8),$$

and

$$p^2 = 1-a^2\beta^2 \quad \ldots\ldots (9),$$

or

$$p^2 = \frac{b^2}{a^2\gamma^2} - \gamma^2 + \frac{b^2}{a^2} \quad \ldots\ldots (10).$$

Since the given point is on the upper disk, and since b is small, p must be between 1 and 0, and γ between $\frac{b}{a}$ and $\left(\frac{b}{a}\right)^{\frac{1}{2}}$; and between those limits we may write, as a sufficient approximation for our purpose,

$$\tan^{-1}\gamma = \gamma = \frac{b}{a}\left[1+\left(\frac{b}{a}\right)^{\frac{1}{2}}\right]\left[p+\left(\frac{b}{a}\right)^{\frac{1}{2}}\right]^{-1} \quad \ldots\ldots (11).$$

We have now to find the value of the surface integral of the product of the density into the potential taken over the upper disk, or

$$\int 2\pi r\, dr\, \sigma\psi = A_1A_2a^{-1}\left(\frac{\pi}{2} - \int_0^1 \tan^{-1}\gamma\, dp\right) \quad \ldots\ldots (12).$$

Substituting the value of $\tan^{-1}\gamma$ from (11), the integral in (12) becomes

$$A_1A_2a^{-1}\left\{\frac{\pi}{2} - \frac{b}{a}\left[1+\left(\frac{b}{a}\right)^{\frac{1}{2}}\right]\log\left[\left(\frac{a}{b}\right)^{\frac{1}{2}}+1\right]\right\} \quad \ldots\ldots (13).$$

The corresponding quantity for the action of the upper disk on itself is got by putting $A_1 = A_2$ and $b = 0$, and is

$$A_2^2 a^{-1} \frac{\pi}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

In the actual case, $A_1 = A_2 = \frac{1}{2}E$, when E is the whole charge; and the lower limit of the capacity is therefore

$$K > \frac{2a}{\pi - \frac{b}{a}\left[1 + \left(\frac{b}{a}\right)^{\frac{1}{2}}\right] \log\left[\left(\frac{a}{b}\right)^{\frac{1}{2}} + 1\right]} \quad \ldots\ldots\ldots\ldots (15);$$

but since we have assumed that b is very small compared with a, we may express our result with sufficient accuracy in the form

$$K = \frac{2}{\pi}\left(a + \frac{1}{2\pi} b \log \frac{a}{b}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16),$$

or, the capacity of two equal disks is equal to that of a single disk whose circumference exceeds that of either disk by $b \log \frac{a}{b}$.

If the space between the disks is filled up, so as to form a single disk of sensible thickness, there will be a certain charge on the cylindric surface; but, at the same time, the charge on the inner sides of the disks will vanish, and that on the outer sides of the disks will be diminished, so that the capacity of a disk of sensible thickness is very little greater than that given by (16).

April 11th, 1878.

C. W. MERRIFIELD, Esq., F.R.S., Vice-President, in the Chair.

Mr. Artemas Martin was elected a Member.

Messrs. W. M. Hicks, M A., Fellow of St. John's College, Cambridge, and T. R. Terry, M.A., Fellow of Magdalen College, Oxford, were proposed for election.

The Chairman, on the recommendation of the Council, nominated Messrs. Brioschi, Darboux, Gordan, Sophus Lie, and Mannheim as Honorary Foreign Members, the Council having decided to raise the number of Foreign Members to twelve.

Mr. Tucker read a letter from Prof. Tait, in which he stated that by a simple *physical* process he could easily obtain any number of definite integrals similar to the following:—

$$\int_0^{\pi}\int_0^{\pi} \sin\theta \, d\phi \, d\theta \times \frac{r^2-a^2-\lambda^2+2a\lambda\cos\theta}{[r^2+a^2+\lambda^2-2a\lambda\cos\theta-2r\lambda\cos\alpha+2ar(\cos\alpha\cos\theta-\sin\alpha\sin\theta\cos\phi)]^{\frac{3}{2}}}$$
$$= \frac{2\pi(r^2-\lambda^2)}{(r^2+\lambda^2-2r\lambda\cos\alpha)^{\frac{3}{2}}}.$$

Prof. Tait wished to know whether the solution could be easily effected by direct analytical processes.

Prof. H. J. S. Smith, F.R.S., Vice-President, then read two papers: Second Notice "On the Characteristics of the Modular Curves," and a Note relating to the "Theory of the Division of the Circle." Prof. Cayley spoke on the subject of both papers, asking, in the course of his remarks, if a solution had been effected for the inscription of a regular heptagon, assuming the trisection of an angle.

Mr. Tucker then read abstracts of papers by Prof. Minchin "On the Astatic Conditions of a Body acted on by given Forces," and by Mr. C. Leudesdorf "On certain Extensions of Frullani's Theorem."

The following presents were received:—

"Journal of Education," April, 1878.

"Elements of Dynamic," Part i., Kinematic, by Prof. W. K. Clifford, F.R.S.; Macmillan, 1878. From the Author.

"Atti della R. Accad. dei Lincei . . . .," Serie terza; "Transunti," Vol. ii., Fasc. 4°, Marzo, 1878; Roma.

"Crelle's Journal," 85 Band, 1^es Heft; Berlin, 1878.

"Jahrbuch über die Fortschritte der Mathematik . . . .," achter Band, Jahrgang, 1876, Heft i.; Berlin, 1878.

"Anvendelse af en Sœtning af Maxwell til at finde de billigste Bygningskonstruktioner af Dr. H. G. Zeuthen."

"Educational Times," April, 1878.

"Journal of Institute of Actuaries," No. cx., Jan. 1878.

"Proceedings of Royal Society," No. 186, Vol. xxvii.

"Monatsbericht," Januar, 1878; Berlin.

On Astatic Equilibrium. *By* Prof. Minchin.

[*Read April 11th*, 1878.]

1. When a body is in equilibrium under the action of forces applied at given points in the body, with fixed magnitudes, and directions fixed in space, it will, under certain conditions, remain in equilibrium when

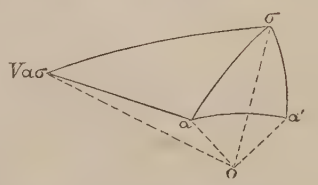
it is displaced in any manner. These are the *Astatic Conditions* of the given forces. They have been investigated at great length in Moigno's "Statique" (Dixième Leçon), and also in a memoir by M. Darboux (Bordeaux, 1877), which has been recently presented to this Society, but which I have not yet thoroughly examined. I believe, however, that several of the results in the following paper have not been noticed by either of these writers.

The method of treatment which I have adopted is one which involves only the very elements of quaternions—the most simple scalar and vector operations, in fact.

2. The equilibrium of a rigid body is preserved by two conditions—namely, the vanishing of the Resultant of Translation of the acting forces, and the vanishing of the Principal Moment calculated for any origin.

3. The general displacement of a rigid body can always be produced by a motion of translation, together with a motion of rotation about some axis, and such is the displacement which must be contemplated in discussing the Astatic Conditions. It is obvious, however, that a motion of translation can alter nothing in the equilibrium of the forces, so that we may confine our attention to a displacement produced by the rotation of the body about an axis.

4. Suppose that a body is rotated about an axis drawn through an origin O in the direction of a unit vector σ, what does any vector, α, drawn from O to a fixed point in the body, become?



Let a sphere, described with O as centre, meet the directions of σ, α, and $V\alpha\sigma$ in points represented in the figure by these letters. Suppose that after rotation the vector α takes the direction denoted by α'; let the angle between $O\alpha$ and $O\sigma$ be θ, and let the body rotate through an angle ϕ. Assume

$$\alpha' = x\alpha + y\sigma + zV\alpha\sigma;$$

$$\therefore \quad S\alpha\alpha' = x\alpha^2 + yS\alpha\sigma,$$

$$S\sigma\alpha' = xS\alpha\sigma + y\sigma^2,$$

$$S\alpha' V\alpha\sigma = z\,(V\alpha\sigma)^2 = -z\alpha^2 \sin^2\theta.$$

Now $\quad S\alpha\alpha' = -\tau^2\alpha\,(\cos^2\theta + \sin^2\theta\cos\phi)$, since $\tau\alpha' = \tau\alpha$;

also $\quad S\sigma\alpha' = -\tau\alpha\,.\cos\theta, \quad S\alpha' V\alpha\sigma = \tau^2\alpha\,.\sin^2\theta\sin\phi.$

Hence we have

$$x = \cos\phi; \quad y = (1-\cos\phi)\,\tau\alpha\,.\cos\theta = -(1-\cos\phi)\,S\alpha\sigma; \quad z = \sin\phi;$$

$$\therefore \quad \alpha' = \alpha\cos\phi - (1-\cos\phi)\,\sigma S\alpha\sigma + \sin\phi V\alpha\sigma,$$

which determines the vector into which α is transformed by rotation.

5. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be the vectors from a fixed origin to the points in the body at which forces represented in magnitudes and directions by the vectors $\varpi_1, \varpi_2, \varpi_3, \ldots$ are applied. Then the resultant of translation is $\varpi_1+\varpi_2+\varpi_3+\ldots$, or $\Sigma\varpi$; and if G is the vector axis of the resultant couple for the assumed origin,

$$G = V(\alpha_1\varpi_1+\alpha_2\varpi_2+\ldots).$$

The conditions of equilibrium are, then,

$$\Sigma\varpi = 0, \quad \Sigma V\alpha\varpi = 0.$$

Now by the rotation of the body about any axis through O, G becomes,

by §4, $\cos\phi\Sigma V\alpha\varpi+2\sin^2\frac{\phi}{2}\Sigma(V\varpi\sigma S\alpha\sigma)-\sin\phi\Sigma(V.\,\varpi V\alpha\sigma)$.

$\Sigma\varpi$, of course, remains unchanged. This expression must vanish independently of ϕ and σ; hence we must have

$$\Sigma(V\varpi\sigma S\alpha\sigma)\equiv 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$\Sigma(V.\,\varpi V\alpha\sigma)\equiv 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

The second of these equations gives

$$\sigma(S\alpha_1\varpi_1+S\alpha_2\varpi_2+\ldots)-(\alpha_1 S\varpi_1\sigma+\alpha_2 S\varpi_2\sigma+\ldots)=0 \ldots\ldots\ldots(3),$$

which, since $\sigma^2=-1$, involves the conditions

$$S\alpha_1\varpi_1+S\alpha_2\varpi_2+\ldots\equiv 0,$$
$$\alpha_1 S\varpi_1\sigma+\alpha_2 S\varpi_2\sigma+\ldots\equiv 0,$$

The first of these expresses that the Virial of the given forces vanishes.

That these conditions must hold may be otherwise seen thus:—Equation (3) may be written $\phi\sigma = g\sigma$, if $\phi\sigma\equiv\Sigma\alpha S\varpi\sigma$, and this equation is, in general, satisfied by only three definite values of σ.

It is not difficult to see that a combination of (1) and (2) gives the equation $\varpi_1 S\alpha_1\sigma+\varpi_2 S\alpha_2\sigma+\ldots\equiv 0$,

or $\phi'\sigma\equiv 0$, where ϕ' is the conjugate of ϕ. But if $\phi\sigma\equiv 0$, $\phi'\sigma$ will also vanish identically, so that the conditions expressed by (1) and (2) may both be represented by the equation $\phi\sigma\equiv 0$.

Again, it will be easily seen that the equations $\Sigma S\alpha\varpi = 0$, $\Sigma V\alpha\varpi = 0$ are both deducible from the equation $\phi\sigma\equiv 0$; so that finally the Astatic conditions are

$$\Sigma\varpi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$
$$\Sigma\alpha S\varpi\sigma\equiv 0.$$

6. The identical vanishing of the linear vector function $\phi\sigma$ will be guaranteed if this function vanishes when any three non-coplanar

vectors i, j, k are substituted for σ. For simplicity, we shall suppose that i, j, k are a rectangular set of unit vectors.

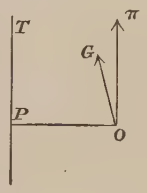
Hence the Astatic Conditions are

$$\Sigma\varpi = 0 \quad \ldots\ldots(5),$$

$$\Sigma a S i\varpi = 0, \quad \Sigma a S j\varpi = 0, \quad \Sigma a S k\varpi = 0 \quad \ldots\ldots(6).$$

7. *To find the equation of Poinsot's Axis.*

For any assumed origin, O, let G be the vector axis of the principal couple, and let Π (or $\Sigma\varpi$) be the Resultant of Translation. Then it is well known that Poinsot's Axis is constructed by taking a point P (in the proper direction) on the right line OP perpendicular to Π and G, at a distance from O equal to $\frac{TG}{T\Pi}\sin\phi$, where ϕ is the angle between G and Π. Hence the vector OP is $\frac{V\Pi G}{T^2\Pi}$; and the equation of Poinsot's Axis is



$$\rho = x\Pi + \frac{V\Pi G}{T^2\Pi}.$$

8. *To find the conditions that a system of forces should be astatically equivalent to a single force.*

Suppose that a force Π_1, acting at the extremity of a vector A_1 drawn to a point fixed in the body, astatically equilibrates the given system.

Denote $\Sigma\varpi$ by Π, and $\Sigma a S i\varpi$, $\Sigma a S j\varpi$, $\Sigma a S k\varpi$ by I, J, K respectively. Then we have, by (5) and (6),

$$A_1 S i\Pi = I, \quad A_1 S j\Pi = J, \quad A_1 S k\Pi = K,$$

which give the conditions

$$\frac{I}{Si\Pi} = \frac{J}{Sj\Pi} = \frac{K}{Sk\Pi}.$$

These conditions are always satisfied when the given forces form a parallel system, since $\frac{I}{Si\Pi}$ obviously becomes $\frac{\Sigma a T\varpi}{T\Pi}$, which is also the common value of $\frac{J}{Sj\Pi}$ and $\frac{K}{Sk\Pi}$, and the vector to the centre of parallel forces.

In general, the vectors $\frac{I}{Si\Pi}, \frac{J}{Sj\Pi}, \frac{K}{Sk\Pi}$ are those drawn from the origin to the points of application, P_i, P_j, P_k, of three systems of parallel forces which are obtained by resolving each force into three rectangular components in the directions i, j, k.

In the present case, then, the points P_i, P_j, P_k must coincide.

9. *To find the conditions that a system of forces should be astatically equivalent to two forces.*

Let the two forces be Π_1 and Π_2 at points whose vectors are A_1 and A_2.

Hence
$$\Pi_1+\Pi_2+\Pi=0 \quad \ldots\ldots (7),$$
$$A_1 Si\Pi_1+A_2 Si\Pi_2=-I \quad \ldots\ldots (8),$$
$$A_1 Sj\Pi_1+A_2 Sj\Pi_2=-J \quad \ldots\ldots (9),$$
$$A_1 Sk\Pi_1+A_2 Sk\Pi_2=-K \quad \ldots\ldots (10).$$

The last three equations show that I, J, K must be coplanar with A_1 and A_2. Hence
$$SIJK=0 \quad \ldots\ldots (11).$$

But there is another condition to be satisfied; for if $lI+mJ+nK\equiv 0$, when l, m, n are given scalars, we have, by multiplying (8), (9), and (10) by l, m, n, and adding,

$$lSi\Pi_1+mSj\Pi_1+nSk\Pi_1=0 \quad \ldots\ldots (12),$$
$$lSi\Pi_2+mSj\Pi_2+nSk\Pi_2=0 \quad \ldots\ldots (13),$$

which, by (7), give the second condition,

$$lSi\Pi+mSj\Pi+nSk\Pi=0 \quad \ldots\ldots (14).$$

Now it is easy to see that (11) and (14) signify that *the three centres, P_i, P_j, P_k of parallel forces are in one right line.* We shall call this the *Line of Centres.* Moreover, *the points of application of Π_1 and Π_2 must also lie on this line.* For (8) can be written

$$A_1 Si\Pi_1+A_2 Si\Pi_2+Si\Pi\,\frac{I}{Si\Pi}=0;$$

and since, by (7), $Si\Pi_1+Si\Pi_2+Si\Pi=0$, this equation shows that the points at which Π_1 and Π_2 act lie on a right line through P_i; similarly, they lie on a right line through P_j; therefore, &c.

Again, the forces Π_1 and Π_2 are obviously known in magnitudes and directions from equations (7) ... (10) as soon as their points of application are assumed; *and in all cases they are both parallel to a given plane;* for (12) gives

$$S\,(li+mj+nk)\,\Pi_1=0,$$

and a similar equation in Π_2, which show that Π_1 and Π_2 are both perpendicular to the vector $li+mj+nk$, as is also the Resultant of Translation of the given system, by (14).

If we assume A_1 and A_2, *i.e.*, take

$$A_1=-\frac{I}{a}+x\left(\frac{I}{a}-\frac{J}{b}\right), \quad A_2=-\frac{I}{a}+y\left(\frac{I}{a}-\frac{J}{b}\right),$$

where, for shortness, we use a, b, c for $-Si\Pi$, $-Sj\Pi$, $-Sk\Pi$, equations (8), (9), (10) give

$$Si\Pi_1 = -\frac{ay}{x-y}, \quad Sj\Pi_1 = -\frac{b(y-1)}{x-y}, \quad Sk\Pi_1 = -\frac{cny+bm}{n(x-y)},$$

$$Si\Pi_2 = \frac{ax}{x-y}, \quad Sj\Pi_2 = \frac{b(x-1)}{x-y}, \quad Sk\Pi_2 = \frac{cnx+bm}{n(x-y)},$$

which of course determine Π_1 and Π_2.

If we take Π_1 and Π_2 at right angles to each other, we have

$$Si\Pi_1 Si\Pi_2 + Sj\Pi_1 Sj\Pi_2 + Sk\Pi_1 Sk\Pi_2 = 0,$$

which gives

$$n^2(a^2+b^2+c^2)xy + bn(cm-bn)(x+y) + b^2(m^2+n^2) = 0.$$

Now it is obvious that the distance of the extremity of A_1 from P_i is $x \,.\, P_iP_j$, and the distance of the extremity of A_2 from P_i is $y \,.\, P_iP_j$; and this last equation shows that these distances (ξ_1 and ξ_2) are connected by an equation of the form

$$\xi_1\xi_2 + p(\xi_1+\xi_2) + q = 0,$$

and that, therefore, the points at which Π_1 and Π_2 are applied are conjugate points of an involution system on the line of centres $P_iP_jP_k$.

The distance of the centre of this involution system from P_i is, of course,

$$\frac{b(bn-cm)}{n(a^2+b^2+c^2)} P_iP_j,$$

so that the vector to this centre is

$$-\frac{I}{a} + \frac{b(bn-cm)}{n(a^2+b^2+c^2)}\left(\frac{I}{a} - \frac{J}{b}\right),$$

or

$$-\frac{aI+bJ+cK}{a^2+b^2+c^2}.$$

Now it is easy to see that this is the vector to the centre of a system of parallel forces whose common direction is that of the Resultant of Translation of the given system.

For, the component of the force ϖ_1 in this direction is $-\frac{S\varpi_1\Pi}{T\Pi}$; therefore the vector of the centre of the parallel system is

$$\frac{a_1 S\varpi_1\Pi + a_2 S\varpi_2\Pi + \ldots}{S\varpi_1\Pi + S\varpi_2\Pi + \ldots},$$

or

$$-\frac{a_1 S(ai+bj+ck)\varpi_1 + a_2 S(ai+bj+ck)\varpi_1 + \ldots}{T^2\Pi},$$

or

$$-\frac{aI+bJ+cK}{a^2+b^2+c^2}.$$

Hence, *when a system of forces is astatically equilibrated by two rectangular forces, the points of application of these latter must lie on the line of centres, and be conjugate points of an involution system whose centre is the centre of a system parallel to the Resultant of Translation.*

Call this point the *centre of the line of centres.*

It remains to be proved that the line of centres is unique. For this purpose we shall show that, if the given forces are each resolved in the direction of any vector, the centre of this system will lie on the line $P_i P_j P_k$.

If i' is the assumed vector, let $i' = xi + yj + zk$; then the vector to the centre of forces parallel to i' is

$$\frac{a_1 Si'\varpi_1 + a_2 Si'\varpi_2 + \dots}{Si'\varpi_1 + Si'\varpi_2 + \dots}, \quad \text{or} \quad -\frac{xI + yJ + zK}{ax + by + cz}.$$

If I' is this vector, we have

$$(ax + by + cz)\, I' - ax\frac{I}{-a} - by\frac{J}{-b} - cz\frac{K}{-c} = 0;$$

and since the sum of the multipliers of I', $-\frac{I}{a}$, $-\frac{J}{b}$, and $-\frac{K}{c}$, is zero, and the extremities of the latter three are collinear, the extremity of I' must lie on the line of centres.

The relations between the vectors A_1 and A_2 and the forces Π_1 and Π_2 will perhaps be better seen if we use Ω for the vector to the centre of the line of centres, and θ for the unit vector in the direction of the line of centres.

Clearly then we may put, when Π_1 and Π_2 are rectangular,

$$\left.\begin{aligned} A_1 &= \Omega + x\theta \\ A_2 &= \Omega - \frac{h^2}{x}\theta \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots (15),$$

$$I = -a\Omega - s\theta, \quad J = -b\Omega - s'\theta, \quad K = -c\Omega - s''\theta,$$

where h^2, s, s', s'' are all given constants, and x any variable scalar.

Equations (8), (9), (10) then determine Π_1 and Π_2, and give

$$\Pi_1 = -\frac{h^2\Pi + x\Theta}{h^2 + x^2}, \quad \Pi_2 = -\frac{x^2\Pi - x\Theta}{h^2 + x^2} \quad \dots\dots\dots\dots\dots\dots (16),$$

where $\Theta = si + s'j + s''k$, so that Θ is a given vector parallel to the plane to which Π, Π_1, and Π_2 are all parallel.

It is easy to prove in different ways that the vectors Π and Θ are perpendicular to each other. One simple method consists in the fact that $S\Pi_1\Pi_2$ must be zero independently of the value of x. It may be seen otherwise thus:

$$aI + bJ + cK = -(a^2 + b^2 + c^2)\,\Omega - (as + bs' + cs'')\,\theta;$$

but

$$\Omega = -\frac{aI + bJ + cK}{a^2 + b^2 + c^2};$$

$\therefore\ as+bs'+cs''=0$, which is the condition that Θ and Π should be perpendicular.

The condition $S\Pi_1\Pi_2=0$ further gives

$$h^2\Pi^2-\Theta^2=0,$$

or, if R be $T\Pi$,

$$T\Theta=hR \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Corresponding to different values of x we can represent the magnitudes and directions of Π_1 and Π_2. Since they equilibrate Π, they must, of course, be represented by the two sides of a right-angled triangle described, in a plane fixed in space, on Π as hypotenuse.

Draw from any origin, O, two right lines, OA and OB, in the directions of Π and Θ respectively. Then, if the components of Π_1 along OA and OB are X_1 and Y_1, we have

$$X_1=\frac{h^2R}{h^2+x^2},\quad Y_1=\frac{hRx}{h^2+x^2};$$

therefore, if $\Pi_1=\overline{OP_1}$, we have

$$\tan P_1OA=\frac{x}{h},$$

which determines the direction of Π_1.

Again, we can prove the following theorem :—

As their points of application along the line of centres vary, the two rectangular forces which astatically equilibrate the system trace out a hyperbolic paraboloid.

For if ρ is the vector to any point on the surface traced out by Π_1,

$$\rho=\Omega+x\theta+y\,\frac{h^2\Pi+x\Theta}{h^2+x^2},$$

where x and y are any variable scalars; or, if the centre of the line of centres is taken as origin of vectors,

$$\rho=x\theta+\frac{h^2y}{h^2+x^2}\,\Pi+\frac{xy}{h^2+x^2}\,\Theta.$$

The Cartesian equation of this surface, referred to the line of centres as axis of x, and those of Π and Θ as axes of y and z, respectively, is

$$xy=hz.$$

Since the two forces Π_1 and Π_2 are equivalent to the given system, the discussion of these forces may replace the discussion of the system. For example, to find Poinsot's Axis, we have, from (15) and (16),

$$VA_1\Pi_1+VA_2\Pi_2=-V(\Omega\Pi+\theta\Theta).$$

Hence, from § 7, Poinsot's Axis is

$$\rho=x\Pi+\Omega-\frac{1}{R^2}\,V\theta\Theta\Pi;$$

or, if the centre of the line of centres is taken as origin of vectors,

$$\rho = x\Pi - \frac{1}{R^2} V\theta\Theta\Pi,$$

so that Poinsot's Axis will coincide with the Resultant of Translation at the centre of the line of centres if $V\theta\Theta\Pi = 0$, or, in other words, if the line of centres is perpendicular to the plane (of Θ and Π) to which the forces Π_1, Π_2, and Π are parallel. The body may, of course, be turned round so that this perpendicularity occurs.

A system of forces can be astatically equilibrated by two forces when all the forces of the system are parallel to one plane.

For let the unit vector perpendicular to this plane be taken as k. Then $Sk\varpi_1 = 0$, $Sk\varpi_2 = 0$, ... ; therefore $K = 0$, and the requisite conditions (11) and (14) are satisfied.

A system of forces may be astatic for displacements about a particular axis without being astatic for displacements about other axes.

If σ is the unit vector in the direction of the axis of displacement, the conditions of continuous equilibrium are

$$\Sigma\varpi = 0,$$

$$\cos\phi\, \Sigma Va\varpi + 2\sin^2\frac{\phi}{2}\, \Sigma\,(V\varpi\sigma Sa\sigma) - \sin\phi\, \Sigma V.\, \varpi Va\sigma = 0;$$

the latter, holding for all values of ϕ, gives

$$\Sigma Va\varpi = 0, \quad \Sigma\,(V\varpi\sigma Sa\sigma) = 0, \quad \Sigma V.\, \varpi Va\sigma = 0.$$

Consider the case in which all the forces lie in one plane, and let the axis of displacement be any axis perpendicular to this plane. Also take the origin of vectors in the plane.

Then these conditions become, since

$$Sa_1\sigma = Sa_2\sigma = ... = 0, \qquad S\varpi_1\sigma = S\varpi_2\sigma = ... = 0,$$
$$\Sigma\varpi = 0, \quad \Sigma Va\varpi = 0, \quad \Sigma Sa\varpi = 0.$$

I have shown ("Statics," pp. 101, 102) how these conditions are otherwise deduced, it being observed that $\Sigma Sa\varpi$ is obviously the Virial of the forces.

Such a system of forces as this can always be equilibrated (for the displacements considered) by a single force. For let the force be Π_1 at the extremity of a vector A_1. Then the conditions are

$$\Pi_1 + \Pi = 0, \qquad A_1\Pi_1 + \Sigma a\varpi = 0;$$

$$\therefore \quad \Pi_1 = -\Pi, \qquad A_1 = \frac{\Sigma a\varpi}{\Pi},$$

the expression for A_1 being evidently a vector, since a_1, ϖ_1, ... and Π are all coplanar.

We thus arrive at the "Centre" of the system, and it is very easy

to prove that (see "Statics," p. 101) this point is characterised by the vanishing of the sum of the moments and of the Virial of the forces about it.

10. *To investigate the astatical equivalence of a system of forces to three forces.*

Let the forces be Π_1, Π_2, Π_3 at the extremities of vectors A_1, A_2, A_3.

Then
$$\Pi_1+\Pi_2+\Pi_3=-\Pi \quad \text{.....(18)},$$
$$A_1 Si\,\Pi_1+A_2 Si\,\Pi_2+A_3 Si\,\Pi_3=-I \quad \text{.....(19)},$$
$$A_1 Sj\,\Pi_1+A_2 Sj\,\Pi_2+A_3 Sj\,\Pi_3=-J \quad \text{.....(20)},$$
$$A_1 Sk\Pi_1+A_2 Sk\Pi_2+A_3 Sk\Pi_3=-K \quad \text{.....(21)}.$$

Now it is well known that the extremities of four vectors, ρ_1, ρ_2, ρ_3, ρ_4, drawn from the same origin, will be coplanar if

$$p_1\rho_1+p_2\rho_2+p_3\rho_3+p_4\rho_4=0,$$

where the scalars p_1, p_2, ... satisfy the equation

$$p_1+p_2+p_3+p_4=0.$$

But, $Si\Pi$ being still denoted by $-a$, &c, (19) can be written

$$A_1 Si\Pi_1+A_2 Si\Pi_2+A_3 Si\Pi_3-a\frac{I}{-a}=0,$$

while (18) gives $\quad Si\Pi_1+Si\Pi_2+Si\Pi_3-a=0.$

Hence the extremities of A_1, A_2, and A_3 lie in a plane containing the point P_i, which has been shown to be the centre of a system of parallel forces obtained by resolving each force parallel to i.

The remaining equations show in the same way that these extremities lie in a plane containing P_j and P_k.

Hence *the points of application of the three forces which astatically equilibrate the system lie in the plane of the three centres* P_i, P_j, P_k; that is, the plane containing the extremities of the vectors $-\frac{I}{a}$, $-\frac{J}{b}$, $-\frac{K}{c}$. This plane we shall call the *Plane of Centres.*

The astatic reduction to three forces is therefore always possible.

The plane of centres can be easily shown to be unique; that is, if the forces be all resolved in any common direction, the centre of the system thus obtained will lie in the plane of P_i, P_j, and P_k. For, let the unit vector in the common direction be $xi+yj+zk$. Then the vector to the new centre is

$$\frac{x\Sigma a Si\varpi+y\Sigma a Sj\varpi+z\Sigma a Sk\varpi}{xSi\Pi+ySj\Pi+zSk\Pi}, \quad \text{or} \quad -\frac{xI+yJ+zK}{ax+by+cz};$$

and by multiplying this vector by $ax+by+cz$, and the vectors $-\frac{I}{a}$, $-\frac{J}{b}$, $-\frac{K}{c}$ by $-ax$, $-by$, $-cz$ respectively, and adding, we get a

result which vanishes identically. Therefore the new centre is in the same plane as the others.

The equation of the plane of centres is, of course,

$$S(aVJK+bVKI+cVIJ)\rho = -SIJK \quad \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Operating on (19), (20), (21) with $S \cdot VA_1A_2$, we have

$$Si\,\Pi_1 \cdot SA_1A_2A_3 = -SIVA_2A_3,$$
$$Sj\,\Pi_1 \cdot SA_1A_2A_3 = -SJVA_2A_3,$$
$$Sk\Pi_1 \cdot SA_1A_2A_3 = -SKVA_2A_3;$$

which give

$$\Pi_1 = \frac{1}{SA_1A_2A_3}(iSIVA_2A_3+jSJVA_2A_3+kSKVA_2A_3);$$

and in the same way we obtain Π_2 and Π_3, so that when their points of application are assumed the forces Π_1, Π_2, Π_3 are thus completely known.

It may be observed that the origin of vectors can always be so chosen that the vectors I, J, K shall be a rectangular system, the vectors i, j, k remaining the same. For the points P_i, P_j, P_k depend only on the vectors i, j, k, and not on the origin of vectors; and, given three points, P_i, P_j, P_k, two other points, O and O', can be found such that the lines OP_i, OP_j, and OP_k are a rectangular system, as are also the lines $O'P_i$, $O'P_j$, and $O'P_k$.

The points O and O' are the points common to three spheres described on the sides of the triangle $P_iP_jP_k$ as diameters; they are equidistant from this plane on opposite sides, and lie on the perpendicular to it drawn through the orthocentre of the triangle.

We may suppose either of these points taken as origin of vectors, and treat I, J, K as a rectangular system.

For simplicity, denoting the vectors VA_2A_3, VA_3A_1, VA_1A_2 by ϵ_1, ϵ_2, ϵ_3, we have

$$\Pi_1 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}(iSI\epsilon_1+jSJ\epsilon_1+kSK\epsilon_1) \quad \ldots\ldots\ldots\ldots (23),$$

$$\Pi_2 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}(iSI\epsilon_2+jSJ\epsilon_2+kSK\epsilon_2) \quad \ldots\ldots\ldots\ldots (24).$$

$$\Pi_3 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}(iSI\epsilon_3+jSJ\epsilon_3+kSK\epsilon_3) \quad \ldots\ldots\ldots\ldots (25).$$

It may be noticed that, for different systems of rectangular vectors i', j', k', the vectors to the centres of the corresponding systems of parallel forces are in the directions of conjugate diameters of a certain ellipsoid. For if $i'=xi+yj+zk$, $j'=x'i+y'j+z'k$, $k'=x''i+y''j+z''k$, it is obvious that I', J', K' are $xI+yJ+zK$, $x'I+y'J+z'K$,

$x''I + y''J + z''K$ respectively. And since $xx' + yy' + zz' = 0$, &c., it follows that

$$\frac{SII'SIJ'}{I^4} + \frac{SJI'SJJ'}{J^4} + \frac{SKI'SKJ'}{K^4} = 0, \quad \&c.,$$

showing that the vectors to new centres are in the directions of conjugate diameters of the ellipsoid

$$\frac{S^2I\rho}{I^4} + \frac{S^2J\rho}{J^4} + \frac{S^2K\rho}{K^4} = 1.$$

The origin of vectors is now a fixed point in the body, and the points P_i, P_j, P_k are, of course, fixed points in the body; and, by the nature of astatic equilibrium, we may consider the body as being placed in any position whatever. Suppose, then, that it is so turned round the origin of vectors that I, J, and K coincide in directions with i, j, and k respectively. This may be regarded as a sort of *initial position* of the body. Let the tensors of I, J, and K be t_1, t_2, and t_3 respectively. Then $iSI\epsilon_1 + jSJ\epsilon_1 + kSK\epsilon_1$ becomes $t_1 iSi\epsilon_1 + t_2 jSj\epsilon_1 + t_3 kSk\epsilon_1$, that is, a self-conjugate linear vector function of ϵ_1. Denote it by $\phi\epsilon_1$. Then (23), (24), and (25) give

$$\Pi_1 = \frac{\phi\epsilon_1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}, \quad \Pi_2 = \frac{\phi\epsilon_2}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}, \quad \Pi_3 = \frac{\phi\epsilon_3}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}} \quad \ldots (26).$$

But it is well known that in the ellipsoid $S\rho\phi\rho = 1$, the normal at the extremity of a vector α is parallel to $\phi\alpha$.

Hence we have the following theorem :—

The body being placed in the initial position, the forces applied at the extremities of any three vectors, A_1, A_2, A_3, drawn from the origin to points in the plane of centres, are in the directions of normals to the ellipsoid $S\rho\phi\rho = 1$ at the points where its surface is intersected by the vectors VA_1A_2, VA_2A_3, VA_3A_1.

If we wish the forces Π_1, Π_2, Π_3 to be a mutually rectangular set, we must take $S\phi\epsilon_1 S\phi\epsilon_2 = 0$, &c., from which it is evident that the directions of ϵ_1, ϵ_2, and ϵ_3 must be conjugate diameters of the ellipsoid

$$S\rho\phi^2\rho = 1.$$

Of course the points P_i, P_j, P_k may be taken as those at which the forces are applied. The forces will then be in the directions i, j, k respectively, and in the initial position they will meet in a point. Their magnitudes are obviously a, b, c.

A case in which the reduction to three forces fails deserves to be noticed.

Suppose the direction of the vector i to be chosen so as to coincide with that of the Resultant of Translation. Then j and k are perpendicular to this direction, and therefore b and c, the sums of the resolved parts of the forces in directions perpendicular to the Resultant of

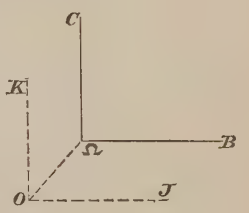
Translation, are each zero. Hence P_j and P_k are at infinity, while P_i is, of course, the centre of the plane of centres. The vectors J and K of course remain, and it is easy to see that the directions of j and k may be so chosen that J and K are perpendicular to each other. For if a new vector j' makes an angle θ with j, we have

$$j' = j\cos\theta + k\sin\theta, \quad k' = -j\sin\theta + k\cos\theta;$$

and since $J' = \Sigma\alpha S j'\varpi$, $K' = \Sigma\alpha S k'\varpi$, it is clear that

$$J' = J\cos\theta + K\sin\theta,$$
$$K' = -J\sin\theta + K\cos\theta,$$

so that J' and K' will be perpendicular if $\tan 2\theta = \dfrac{2SJK}{K^2 - J^2}$.



Let Ω in the figure be the centre of the plane of centres; OJ and OK the vectors J and K drawn through the origin of vectors, O; and let ΩB and ΩC be drawn (in the plane of centres) parallel to J and K.

Then, if $O\Omega$ be the perpendicular to this plane at Ω, it is clear that the points P_i, P_j, P_k subtend right angles in pairs at O, the latter two being at infinity on ΩB and OC respectively.

Suppose that the force Π_1 is applied at Ω, *i.e.*, $A_1 = \overline{O\Omega} = -\dfrac{I}{a}$. Then, the body being in its initial position, the axes of the ellipsoids $S\rho\phi\rho = 1$ and $S\rho\phi^2\rho = 1$ are in the directions $O\Omega$, OJ, and OK. Hence the vectors ϵ_2 and ϵ_3 are in the plane JOK, and (the applied forces being supposed a rectangular system) ϵ_1 coincides in direction with A_1; therefore the vectors A_2 and A_3 are in the plane JOK, and their extremities are therefore at infinity. The forces Π_2 and Π_3 are then applied at infinity, and we can see that their magnitudes are each zero. For, since $-S\epsilon_1\epsilon_2\epsilon_3 = S^2 A_1 A_2 A_3$, it is very easy to show in equations (26) that

$$T\Pi_1 = \frac{1}{TA_1 \,.\, \sin p_1}\sqrt{t_1^2\cos^2\theta_1 + t_2^2\cos^2\phi_1 + t_3^2\cos^2\psi_1},$$

where p_1 is the angle between the direction of A_1 and the plane of A_2A_3, and θ_1, ϕ_1, ψ_1 are the angles between ϵ_1 and the directions of I, J, K. Similar values are obtained for $T\Pi_2$ and $T\Pi_3$.

Now, in the present case, $TA_2 = TA_3 = \infty$; therefore Π_2 and Π_3 are zero forces applied at infinity.

This result, of course, indicates a new mode of reduction—namely, to a force and two couples; and this is the mode of reduction adopted in all cases by Moigno.

Let the centre of the plane of centres be taken as origin of vectors, and suppose a force Π_2 applied at the extremity of a vector γ, while a force $-\Pi_2$ is applied at the extremity of γ'. Then in equa-

tion (19) we shall have the term $\gamma Si\Pi_2 - \gamma' Si\Pi_2$, or $(\gamma - \gamma')\, Si\Pi_2$. Denote $\gamma - \gamma'$ by μ; then μ is the vector joining the points at which the forces $(\Pi_2, -\Pi_2)$ constituting the couple act. Similarly, let μ' be the vector joining the points at which the forces $(\Pi_3, -\Pi_3)$ act. We may, for shortness, call μ and μ' the *vector arms* of the couples. Then our equations are (I and A_1 being zero)

$$\Pi_1 + \Pi = 0,$$
$$\mu Si\Pi_2 + \mu' Si\Pi_3 = 0,$$
$$\mu Sj\Pi_2 + \mu' Sj\Pi_3 = -J,$$
$$\mu Sk\Pi_2 + \mu' Sk\Pi_2 = -K.$$

The second requires $Si\Pi_2 = 0$, $Si\Pi_3 = 0$; *i.e.*, the forces of the couples are in a plane perpendicular to the Resultant of Translation.

Suppose the body placed in the initial position; then $J = t_2 j$, $K = t_3 k$, and i is the unit vector perpendicular to the plane of centres. Hence we have

$$\mu Sj\Pi_2 + \mu' Sj\Pi_3 = -t_2 j,$$
$$\mu Sk\Pi_2 + \mu' Sk\Pi_3 = -t_3 k.$$

Operate on these with $S \cdot Vi\mu'$, and observe that

$$S\mu Vi\mu' = -Si\mu\mu' = -SiV\mu\mu' = TV\mu\mu';$$

also
$$SjVi\mu' = -Sk\mu';$$

then we have
$$Sj\Pi_2 = \frac{t_2 Sk\mu'}{TV\mu\mu'},\quad Sk\Pi_2 = -\frac{t_3 Sj\mu'}{TV\mu\mu'}.$$

Hence
$$\Pi_2 = \frac{1}{TV\mu\mu'}(-t_2 jSk\mu' + t_3 kSj\mu') \quad \ldots\ldots\ldots\ldots (27),$$

$$\Pi_3 = \frac{1}{TV\mu\mu'}(-t_2 jSk\mu + t_3 kSj\mu) \quad \ldots\ldots\ldots\ldots\ldots (28).$$

Let Π_1 and Π_3 be perpendicular to each other. Then

$$\frac{Sj\mu\, Sj\mu'}{t_2^2} + \frac{Sk\mu\, Sk\mu'}{t_3^2} = 0,$$

which shows that the directions of μ and μ' are those of a pair of conjugate diameters of the ellipse

$$S\rho\, \phi\rho = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29),$$

where $\phi\rho \equiv f^2 \left(\frac{j}{t_2^2} Sj\rho + \frac{k}{t_3^2} Sk\rho \right)$, and f (denoting any constant force magnitude) is introduced for homogeneity.

Assume the arms to be represented, not only in directions, but also in magnitudes, by a pair of semi-conjugate diameters. Then $TV\mu\mu'$ is constant and equal to $\frac{t_2 t_3}{f^2}$, the product of the semiaxes.

Hence, from (27) and (28),

$$\Pi_2 = f^2\left(-j\frac{Sk\mu'}{t_3}+k\frac{Sj\mu'}{t_2}\right)=f^2\left(j\frac{Sj\mu}{t_2}+k\frac{Sk\mu}{t_3}\right),$$

$$\Pi_3 = f^2\left(-j\frac{Sk\mu}{t_3}+k\frac{Sj\mu}{t_2}\right)=f^2\left(j\frac{Sj\mu'}{t_2}+k\frac{Sk\mu'}{t_3}\right).$$

Now the ellipse (29) can be written $T\psi\rho = 1$, where

$$\psi\rho = f\left(j\frac{Sj\rho}{t_2}+k\frac{Sk\rho}{t_3}\right),$$

and $\psi\rho$ obviously denotes the vector to the corresponding point on the circumscribed circle. Hence we have simply

$$\Pi_2 = f.\psi\mu,\quad \Pi_3 = f.\psi\mu' \quad \text{.....(30),}$$

and we arrive at the following result:—

The body having been placed so that the plane of centres is perpendicular to the Resultant of Translation, and the vectors J and K, fixed in the body, coincide with the corresponding vectors j and k fixed in space, the system may be astatically equilibrated by a single force acting at the centre of the plane of centres, equal and opposite to the Resultant of Translation, together with two couples in this plane, the forces of these couples acting in two rectangular directions at the extremities of any pair of semi-conjugate diameters of a certain ellipse, their forces being all equal and of constant magnitude whatever pair of diameters be chosen, and the forces at the extremities of each semi-diameter of the ellipse being parallel to the corresponding semi-diameter of its circumscribed circle.

The force and two couples which astatically equilibrate the system will in certain positions reduce to a single resultant force, and we propose to show that, when this is the case, the line of action of this force intersects two conics fixed in the body.

Choose the arms of the couples in the directions of the axes of the ellipse just mentioned, and let their forces be applied at the extremities of the semi-axes. Then, the body being in any position, the vectors j and k will not coincide with J and K, and we must put

$$A_2 = \frac{J}{f},\quad A_3 = \frac{K}{f},\quad \Pi_2 = fj,\quad \Pi_3 = fk,\quad \Pi_1 = ai.$$

Hence, if G is the vector axis of the resultant couple,

$$G = V(Jj+Kk),$$

and, for a single resultant, $S\Pi G = 0$, or

$$SJk = SKj \quad \text{.....(31).}$$

Again, Poinsot's Axis will coincide with the line of action of the single resultant, and its equation is

$$\rho = xi + \frac{1}{a}(jSiJ + kSiK),$$

(by 7,) or it can evidently be written

$$\rho = \lambda i + \frac{1}{a} V(Jk - Kj) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32),$$

where λ is an arbitrary scalar.

If we denote VJK by I, and TJ, TK, as before, by t_2 and t_3, and also if the direction angles of the vectors I, J, K with respect to i, j, k be represented by the following scheme :—

Equation (31) gives

	i	j	k
I	α_1	β_1	γ_1
J	α_2	β_2	γ_2
K	α_3	β_3	γ_3

$$t_2 \cos\gamma_2 = t_3 \cos\beta_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33).$$

Also, operating on (32) successively with $S.I$, $S.J$, and $S.K$, and denoting $SI\rho$ by $-t_2 t_3 x$, $SJ\rho$ by $-t_2 y$, $SK\rho$ by $-t_3 z$, we have

$$x = \lambda \cos\alpha_1 + \frac{1}{a}(t_3 \cos\beta_3 + t_2 \cos\gamma_3),$$

$$y = \lambda \cos\alpha_2 - \frac{t_3}{a} \cos\beta_1,$$

$$z = \lambda \cos\alpha_3 - \frac{t_2}{a} \cos\gamma_1.$$

To find the point where Poinsot's Axis meets the plane of IJ, put $z=0$; therefore $\lambda = \frac{t_2}{a}\frac{\cos\gamma_1}{\cos\alpha_3}$. Substituting this in the values of x and y, and remembering that I, J, K and i, j, k are two rectangular systems, we shall find, with the aid of (33),

$$\frac{x^2}{t_3^2} + \frac{y^2}{t_3^2 - t_2^2} = \frac{1}{a^2}.$$

Similarly, for the point in which Poinsot's Axis intersects the plane KI,

$$\frac{x^2}{t_2^2} + \frac{z^2}{t_2^2 - t_3^2} = \frac{1}{a^2}.$$

Now these are the focal conics of the quadric

$$\frac{x^2}{t_2^2 + t_3^2} + \frac{y^2}{t_3^2} + \frac{z^2}{t_2^2} = \frac{1}{a^2}.$$

Hence they are both intersected by the line of action of the single resultant to which the system of forces reduces in certain positions of the body.

This is Minding's Theorem.

On certain Extensions of Frullani's Theorem.

By C. Leudesdorf, M.A.

[*Read April 11th*, 1878.]

The following investigation is intended to be supplementary to two papers by Mr. Elliott, on "Certain Multiple Definite Integrals," published in No. 106 and No. 113 of the *Proceedings*. Starting with an amended form of the theorem generally called Frullani's, the author deduces certain elegant extensions of this theorem to the cases of integrals of the 2nd, 3rd, ... 6th order; and in the second paper (§ 5) he writes down (though without proof) the theorem for the general case of the n-tuple integral to which he is led by his results in the cases investigated. These results are shewn to be true when certain conditions hold good among the involved constants; but these conditions are unsymmetrical in form, and their number in any given case seems subject to no law. For the integrals of order 1, 2, ... 6, the number of constants involved is 2, 4, ... 12, while the number of conditions found is 0, 1, 2, 4, 5, 7. There is thus no means of determining, for the general case, how many conditions must hold among the $2n$ constants, nor yet what these conditions are; so that there is no guarantee that it shall be possible to satisfy the conditions with the disposable constants. It is however difficult, on reading Mr. Elliott's papers, to avoid concluding that he is correct in asserting that the general theorem *does* hold, and the following proof of it may perhaps be of interest, as the subject is considered from rather a different point of view from that taken in the two papers mentioned; while the conditions found are symmetrical in form and less than the number of disposable constants, being $n-1$ in number when there are $2n$ constants.

As in the second paper, let $S(p, q, r \ldots)$ denote any symmetric function of $p, q, r \ldots$ which does not become infinite for any positive values of $p, q, r \ldots$ from zero to infinity, both inclusive. Denote

$$\int_0^\infty \frac{\phi(ax)}{x}\,dx \text{ by } [a], \quad \int_0^\infty\int_0^\infty S(ax, by)\,\frac{dx\,dy}{xy} \text{ by } [ab],$$

$$\int_0^\infty\int_0^\infty\int_0^\infty S(ax, by, cz)\,\frac{dx\,dy\,dz}{xyz} \text{ by } [abc], \text{ \&c.}$$

Let also $\log a$, $\log b$, $\log c$ ... be denoted by α, β, γ .... Then Frullani's Theorem in Mr. Elliott's form may be written

$$[\alpha]-[b] = (\alpha-\beta)\{\phi(\infty)-\phi(0)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

By two applications of this,

$$\int_0^\infty\int_0^\infty \{S(ax, cy)-S(bx, cy)-S(ax, dy)+S(bx, dy)\}\frac{dx\,dy}{xy}$$

becomes first

$$(\alpha-\beta)\int_0^\infty \{S(\infty, cy)-S(0, cy)-S(\infty, dy)+S(0, dy)\}\frac{dy}{y},$$

and finally

$$(\alpha-\beta)(\gamma-\delta)\{S(\infty, \infty)-2S(\infty, 0)+S(0, 0)\},$$

which result may be written

$$\begin{aligned}(\alpha-\beta)(\gamma-\delta)\,S(\infty-0)^2 &= [ac]-[bc]-[ad]+[bd]\\ &= [(a-b)(c-d)] \quad \ldots\ldots\ldots\ldots\ldots (2),\end{aligned}$$

by an extension of the notation already used. Similarly

$$[(a-b)(c-d)(e-f)]$$

or
$$\int_0^\infty\int_0^\infty\int_0^\infty \{S(ax, cy, ez)-S(bx, cy, ez)+\ldots\}\frac{dx\,dy\,dz}{xyz}$$
$$= [ace]-[bce]-[ade]+[bde]$$
$$-[acf]+[bcf]+[adf]-[bdf]$$

$$= (\alpha-\beta)$$
$$\times\int_0^\infty\int_0^\infty \left\{\begin{array}{l} S(\infty, cy, ez)-S(0, cy, ez)-S(\infty, dy, ez)+S(0, dy, ez)\\ -S(\infty, cy, fz)+S(0, cy, fz)+S(\infty, dy, fz)-S(0, dy, fz)\end{array}\right\}\frac{dy\,dz}{yz}$$
$$= (\alpha-\beta)(\gamma-\delta)$$
$$\times\int_0^\infty \left\{\begin{array}{l} S(\infty, \infty, ez)-S(\infty, 0, ez)-S(0, \infty, ez)+S(0, 0, ez)\\ -S(\infty, \infty, fz)+S(\infty, 0, fz)+S(0, \infty, fz)-S(0, 0, fz)\end{array}\right\}\frac{dz}{z}$$
$$= (\alpha-\beta)(\gamma-\delta)(\epsilon-\zeta)$$
$$\times\{S(\infty, \infty, \infty)-3S(\infty, \infty, 0)+3S(\infty, 0, 0)-S(0, 0, 0)\}$$
$$= (\alpha-\beta)(\gamma-\delta)(\epsilon-\zeta)\,S(\infty-0)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

And, in the same way, it may be shewn that

$$\begin{aligned}[(a_1-b_1)(a_2-b_2)\ \ldots\ (a_n-b_n)]&\\ = (a_1-\beta_1)(a_2-\beta_2)\ \ldots\ (a_n-\beta_n)\,S(\infty-0)^n \ \ldots\ldots (4).&\end{aligned}$$

And if, by another extension of the notation used, $[a\bar{b}]$ be employed to denote $\int_0^\infty S(ax, b)\frac{dx}{x}$, $[a\bar{b}\bar{c}]$ to denote $\int_0^\infty S(ax, b, c)\frac{dx}{x}$, $[abc]$ to

denote $\int_0^\infty \int_0^\infty S(ax, by, c)\, \frac{dx\,dy}{xy}$, &c., it is seen that

$$[a(b-c)] = (\beta-\gamma)[a\overline{\infty-0}],$$
$$[ab(c-d)] = (\gamma-\delta)[ab\infty-\overline{0}],$$
$$\text{&c.} \qquad \text{&c.},$$

$$[a(b-c)(d-e)] = (\beta-\gamma)(\delta-\epsilon)\int_0^\infty \{S(ax,\infty,\infty)-2S(ax,\infty,0)+S(ax,0,0)\}\,\frac{dx}{x}$$
$$= (\beta-\gamma)(\delta-\epsilon)[a(\overline{\infty-0}^2]$$
$$[a(b-c)(d-e)(f-g)] = (\beta-\gamma)(\delta-\epsilon)(\zeta-\eta)[a\overline{\infty-0}^3], \text{ &c. &c. ..... (5).}$$

Taking now the case of the integral of the second order,

$$[ab]-[a'b'] = [\{r-(r-a)\}\{r-(r-b)\}]-[\{r-(r-a')\}\{r-(r-b')\}]$$

(introducing a new quantity r whose logarithm is ρ)

$$= [rr-r(r-a+r-b)+(r-a)(r-b)] - [rr-r(r-a'+r-b')+(r-a')(r-b')]$$
$$= [(r-a)(r-b)]-[(r-a')(r-b')] + [r(a-a'+b-b')],$$
$$= \{(\rho-\alpha)(\rho-\beta)-(\rho-\alpha')(\rho-\beta')\}\,S(\infty-0)^2 + (\alpha+\beta-\alpha'-\beta')[r\overline{\infty-0}], \text{ by (2) and (5),}$$
$$= (\alpha\beta-\alpha'\beta')\,S(\infty-0)^2,$$

if only $\alpha+\beta = \alpha'+\beta'$.

Taking next the triple integral,

$$-[abc]+[a'b'c'] = [(r-a)(r-b)(r-c)]-[(r-a')(r-b')(r-c')] - [r\{(r-a)(r-b)+(r-b)(r-c)+(r-c)(r-a)\} - r\{(r-a')(r-b')+(r-b')(r-c')+(r-c')(r-a')\}] + [rr(a+b+c-a'-b'-c')]$$
$$= \{(\rho-\alpha)(\rho-\beta)(\rho-\gamma)-(\rho-\alpha')(\rho-\beta')(\rho-\gamma')\}\,S(\infty-0)^3 - \{(\rho-\alpha)(\rho-\beta)+(\rho-\beta)(\rho-\gamma)+(\rho-\gamma)(\rho-\alpha)\}[r\overline{\infty-0}^2] - (\rho-\alpha')(\rho-\beta')-(\rho-\beta')(\rho-\gamma')-(\rho-\gamma')(\rho-\alpha') + (\alpha+\beta+\gamma-\alpha'-\beta'-\gamma')[rr\overline{\infty-0}], \text{ by (3) and (5).}$$

But

$$(\rho-\alpha)(\rho-\beta)(\rho-\gamma)-(\rho-\alpha')(\rho-\beta')(\rho-\gamma') = -\rho^2(\Sigma\alpha-\Sigma\alpha')+\rho(\Sigma\alpha\beta-\Sigma\alpha'\beta')-(\alpha\beta\gamma-\alpha'\beta'\gamma'),$$

$$\left.\begin{aligned}(\rho-\alpha)(\rho-\beta)+(\rho-\beta)(\rho-\gamma)+(\rho-\gamma)(\rho-\alpha)\\ -(\rho-\alpha')(\rho-\beta')-(\rho-\beta')(\rho-\gamma')-(\rho-\gamma')(\rho-\alpha')\end{aligned}\right\} = 2\rho(\Sigma\alpha-\Sigma\alpha')-(\Sigma\alpha\beta-\Sigma\alpha'\beta'),$$

$$\alpha+\beta+\gamma-\alpha'-\beta'-\gamma' = \Sigma\alpha-\Sigma\alpha';$$

so that

$$[abc]-[a'b'c'] = \{\rho^2(\Sigma a-\Sigma a')-\rho(\Sigma a\beta-\Sigma a'\beta')+(a\beta\gamma-a'\beta'\gamma')\}\, S(\infty-0)^3 + \{2\rho^2(\Sigma a-\Sigma a')-\rho(\Sigma a\beta-\Sigma a'\beta')\}\,[r\overline{\infty-0}^2] - \{\rho^2(\Sigma a-\Sigma a')\}\,[r^2\overline{\infty-0}];$$

and when $\Sigma a = \Sigma a'$ and $\Sigma a\beta = \Sigma a'\beta'$,

$$[abc]-[a'b'c'] = (a\beta\gamma-a'\beta'\gamma')\, S(\infty-0)^3.$$

The cases of the double and the triple integral only have been taken for simplicity; but it is clear that precisely similar investigations will hold good for the cases of the integrals of order 4, 5, ... n. The general theorem may be thus enunciated:—

Denote the sums of the homogeneous products of orders 1, 2, ... n of the n quantities $\log\frac{r}{a_1}$, $\log\frac{r}{a_2}$, ... $\log\frac{r}{a_n}$ by Σ_1, Σ_2, ... Σ_n; and let Σ'_1, Σ'_2, ... Σ'_n denote the same with regard to the n quantities $\log\frac{r}{a'_1}$, $\log\frac{r}{a'_2}$, ... $\log\frac{r}{a'_n}$; then

$$(-1)^n\{[a_1a_2\dots a_n]-[a'_1a'_2\dots a'_n]\} = (\Sigma_n-\Sigma'_n)\, S(\infty-0)^n - (\Sigma_{n-1}-\Sigma'_{n-1})\,[r\overline{\infty-0}^{n-1}] + (\Sigma_{n-2}-\Sigma'_{n-2})\,[r^2\overline{\infty-0}^{n-2}] - \&c. + (-1)^n(\Sigma_1-\Sigma'_1)\,[r^{n-1}\overline{\infty-0}];$$

and, just as in the cases of $n=2$, $n=3$, it is seen that $\Sigma_n-\Sigma'_n$, $\Sigma_{n-1}-\Sigma'_{n-1}$, ... can each be expressed in terms of $\sigma_n-\sigma'_n$, $\sigma_{n-1}-\sigma'_{n-1}$, ..., where σ_r stands for the sum of the homogeneous products of order r of the n quantities $\log a_1$, $\log a_2$, ... $\log a_n$, and σ'_r similarly with regard to $\log a'_1$, $\log a'_2$, ... $\log a'_n$. So that, when the $n-1$ conditions $\sigma_1=\sigma'_1$, $\sigma_2=\sigma'_2$, ... $\sigma_{n-1}=\sigma'_{n-1}$ are satisfied,

$$\begin{aligned}[a_1a_2\dots a_n]-[a'_1a'_2\dots a'_n] &= (-1)^n(\Sigma_n-\Sigma'_n)\, S(\infty-0)^n\\ &= (-1)^n S(\infty-0)^n \left\{\begin{array}{l}(\rho-\alpha_1)(\rho-\alpha_2)\dots(\rho-\alpha_n)\\ -(\rho-\alpha'_1)(\rho-\alpha'_2)\dots(\rho-\alpha'_n)\end{array}\right\}\\ &= (-1)^n S(\infty-0)^n \left\{\begin{array}{l}-\rho^{n-1}(\sigma_1-\sigma'_1)\\ +\rho^{n-2}(\sigma_2-\sigma'_2)\\ \quad -\&c.\\ +(-1)^n(\sigma_n-\sigma'_n)\end{array}\right\}\\ &= (\sigma_n-\sigma'_n)\, S(\infty-0)^n;\end{aligned}$$

or

$$\int_0^\infty\int_0^\infty\dots\{S(a_1x_1, a_2x_2, \dots a_nx_n)-S(a'_1x_1, a'_2x_2, \dots a'_nx_n)\}\frac{dx_1dx_2\dots dx_n}{x_1x_2\dots x_n} = (\log a_1\log a_2\dots\log a_n-\log a'_1\log a'_2\dots\log a'_n)\, S(\infty-0)^n,$$

in which form Mr. Elliott enunciates the result in a Note added to his second paper.

Referring to the first paper, it is easily seen that a precisely similar theorem holds good with regard to the integrals there considered; in fact, that

$$\int_0^\infty \int_0^\infty \dots \{\phi(a_1x_1+a_2x_2+\dots+a_nx_n)-\phi(a_1'x_1, a_2'x_2, \dots a_n'x_n)\} \frac{dx_1\,dx_2 \dots dx_n}{x_1x_2 \dots x_n}$$
$$= (-1)^{n-1}(\log a_1 \log a_2 \dots \log a_n - \log a_1' \log a_2' \dots \log a_n')\{\phi(\infty)-\phi(0)\}.$$

This result is of course really included in the one proved above; but it may be established independently of this by an exactly similar method; the fundamental theorem, corresponding to (4), being in this case

$$[(a_1-b_1)(a_2-b_2) \dots (a_n-b_n)]$$
$$= (-1)^{n-1}(\alpha_1-\beta_1)(\alpha_2-\beta_2) \dots (a_n-\beta_n)\{\phi(\infty)-\phi(0)\}.$$

$[a]$ standing for $\int_0^\infty \phi(ax)\frac{dx}{x}$, $[ab]$ for $\int_0^\infty \int_0^\infty \phi(ax+by)\frac{dx\,dy}{xy}$, &c.

May 9th, 1878.

Lord RAYLEIGH, F.R.S., President, in the Chair.

Messrs. W. M. Hicks and T. R. Terry were elected Members, and Prof. G. M. Minchin was admitted into the Society.

Messrs. Brioschi, Darboux, Gordan, Sophus Lie, and Mannheim were elected Honorary Foreign Members.

Prof. Henrici, F.R.S., communicated a paper by Dr. Klein, of Munich, "Uber die Transformation der elliptischen Functionen." Prof. Cayley, F.R.S., gave an account of a paper "On the Theory of Groups." Prof. Kennedy read his "Notes on the Solution of Statical Problems connected with Linkwork, and other Plane Mechanisms." Mr. J. W. L. Glaisher, F.R.S., briefly gave an account of his "Generalized Form of certain Series." Mr. A. B. Kempe spoke on "Conjugate Four-piece Linkages."

The following presents were made to the Society:—

"Aperçu historique sur l'origine et le développement des méthodes en géométrie," par M. Chasles, 2nd edition, 1875. From the Author.

"Sur les écarts de la ligne géodésique et des sections planes normales entre deux points rapprochés d'une surface courbe," par F. J. van de

Berg. (Harlem, extrait des Archives Néerlandaises, t. xii., pp. 353—398.)

"Monatsbericht," Feb. 1878.

"Educational Times," May, 1878.

"Atti della R. Accademia dei Lincei," Anno cclxxv., Serie terza.

"Transunti," Vol. ii., Fasc. 5°, Aprile 1878.

"Memorie della Classe di Scienze Fisiche, Matematiche e Naturali," Vol. i., dispensa 1ma, 2a; Roma, 1877.

"Memorie della Classe di Scienze Morali, Storiche e Filologiche," Vol. i.; Roma, 1877.

"Memorie della R. Accademia di Scienze, Lettere ed Arti in Modena," Tom. xvii.; Modena, 1877.

"Annali di Matematica," Tom. ix., Fasc. i.; Marzo 1878.

"Verallgemeinerung des Gaussischen Criterium für den quadratischen Rest-character einer Zahl in Bezug auf eine andere." Hr. Ernst Schering. (Extract from "Monatsbericht," 22 Juni, 1876.)

Dr. Kronecker, "Ueber Abelsche Gleichungen" (read April 16), Dec. 1877; "Notiz über Potenzreihen," Jan. 21, 1878; "Ueber Sturmsche Functionen," Feb. 14, 1878. (Extracts from "Monatsbericht.")

"Die fundamentalen Anzahlen und Ausartungen der cubischen Plancurven nullten Geschlechts," von H. Schubert in Hamburg. ("Math. Annalen." Bd. xiii., zweite Abhandlung der "Beiträge zur Abzählenden Geometrie.")

"Die Bestimmung und Ausgleichung der aus Beobachtungen abgeleiteten Wahrscheinlichkeiten," von Wilhelm Lazarus in Hamburg.

On the Transformation of Elliptic Functions.

By Dr. F. Klein.*

[*Read May 9th*, 1878.]

My investigations into the solution of equations of the fifth degree have led me to some results concerning the Transformation of Elliptic Functions which I wish to lay before the Mathematical Society. A full discussion of these will soon appear in the "Mathematische Annalen," t. xiv.

Let $\int \frac{dx}{\sqrt{a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4}}$ be an elliptical integral.

We have then the invariants

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2,$$

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* The Secretaries are indebted to Prof. Henrici for the following translation.

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix},$$

and $$\Delta = g_2^3 - 27g_3^2$$

is the value of the discriminant.

As *absolute* invariant I shall take the quotient $\frac{g_2^3}{\Delta}$, and denote it by J,

so that we have $$J = \frac{g_2^3}{\Delta}, \quad J-1 = \frac{27g_3^2}{\Delta}$$

I now consider, as F. Müller and Brioschi have done,* the equations which exist between the absolute invariants J and J' of two elliptic integrals of which one is changed into the other by a transformation of order n, where n denotes a prime number >3, and determine, in particular, the ramification of the corresponding Riemann surface, considering J' as a function of J. It can be shown that branch-points exist only at $J = \infty, 0, 1$. At $J = \infty$ there are n of the $n+1$ sheets of the surface cyclically connected, whilst one remains isolated. For $J = 0$ we have to distinguish whether n is of the form $6\mu+5$ or $6\mu+1$. In the first case all the sheets are at $J = 0$ connected in cycles of three. In the other case two sheets remain isolated, whilst the others are connected in cycles of three. The connection at $J=1$ is similar. If n is of the form $4\nu+3$, all the sheets are connected in pairs. But if n is of the form $4\nu+1$, two sheets are left isolated, whilst the remaining ones are connected in pairs.

I now calculate the deficiency of the equation existing between J and J'. The result shows that the deficiency is $=0$ for $n=5, 7, 13$, and for these values only. In these cases it is therefore possible to express J and J' as rational functions of a parameter, and here it is very remarkable *that these rational functions are perfectly determined by the nature of ramification above explained, so that it is not necessary to return to the elliptic integrals or to the double periodical functions in order to determine them.*

As an example, I explain the case $n=7$, to which the communication relates which I enclose.† Let τ be the parameter in terms of which everything can be expressed rationally, and let us write

$$J = \frac{\phi(\tau)}{\psi(\tau)}$$

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* *F. Müller*, "De Transformatione functionum ellipticarum." Berlin. 1867. *Brioschi*, "Sur une formule de transformation des fonctions elliptiques." "Comptes Rendus." 1875.

† "Ueber Gleichungen siebenten Grades." Berichte der Erlangen Societät. Sitzung vom 4 März, 1878.

where ϕ, ψ are integral functions of the eighth degree. According to what precedes, ψ must consist of a linear and a sevenfold factor. As we might introduce for τ a linear function of τ, we may assume $\psi = c\tau$, where the sevenfold factor has been put equal to a constant. As 7 is of the form $6\mu+1$, it follows that ϕ must have two single and two threefold factors; hence we assume,

$$\phi = (\tau^2+a\tau+\beta)\,(\tau^3+A\tau+B)^3.$$

As τ has been determined, except as to a constant factor, the value of one of the constants may be assumed arbitrarily; for instance, $\beta = 49$, as has been found to be convenient.

Let us next consider $$J-1 = \frac{\phi-\psi}{\psi}.$$

As 7 is of the form $4\nu+3$, it follows that $J-1$ must vanish twice for each of four different values of τ, that is, $(\phi-\psi)$ has to be the square of an expression of the fourth degree. This expression can at once be found. For it has to be a single factor of the functional determinant,

$$\begin{vmatrix} \phi & \psi \\ \frac{\delta\phi}{\delta\tau} & \frac{\delta\psi}{\delta\tau} \end{vmatrix},$$

and this reduces to the fourth degree when the factor $(\tau^2+A\tau+B)^2$ is thrown out. The expression of the fourth degree to a numerical factor is thus found to be

$$(\tau^2+a\tau+49)\,(\tau^2+A\tau+B)-\tau\,(\tau^2+a\tau+49)\,(6\tau+3A) \\ -\tau\,(\tau^2+A\tau+B)\,(2\tau+a).$$

If we now identify the square of this expression with $\phi-\psi$, multiplied by a constant factor, we obtain a more than sufficient number of equations for the determination of a, A, B. These yield after a short calculation

$$J : J-1 : 1 = (\tau^2+13\tau+49)\,(\tau^2+5\tau+1)^3 \\ : (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 \\ : 1728\tau.$$

In order to express J' in terms of τ, I introduce a new parameter τ' by assuming

$$J' : J'-1 : 1 = (\tau'^2+13\tau'+49)\,(\tau'^2+5\tau'+1)^3 \\ : (\tau'^4+14\tau'^3+63\tau'^2+70\tau'-7)^2 \\ : 1728\tau',$$

and seek for the relation between τ and τ'. This must be linear, as τ and τ' are one-valued in the same Riemann surface. By considerations in themselves very simple, but to explain which would lead me too far, it follows that $$\tau\tau' = 49.$$

This equation, together with the two preceding ones, expresses in the

simplest form the relation between J and J' for a transformation of the seventh order.

Similarly, we get, for $n = 5$,

$$\begin{cases} J : J-1 : 1 = (\tau^2-10\tau+5)^3 : (\tau^2-22\tau+125)(\tau^2-4\tau-1)^2 : -1728\tau, \\ \text{the same for } J' \text{ in } \tau', \qquad \tau\tau' = 125 ; \end{cases}$$

and for $n = 13$,

$$\begin{cases} J : J-1 : 1 = (\tau^2+5\tau+13)(\tau^4+7\tau^3+20\tau^2+19\tau+1)^3 \\ \qquad\qquad : (\tau^2+6\tau+13)(\tau^6+10\tau^5+46\tau^4+108\tau^3+122\tau^2+38\tau-1)^2 \\ \qquad\qquad : 1728\tau ; \\ \text{the same for } J' \text{ in } \tau', \qquad \tau\tau' = 13. \end{cases}$$

These equations may, however, be easily solved transcendentally. If we write for any value of n,

$$M = \frac{q^{\frac{n}{6}}\Pi(1-q^{2\nu n})^2}{q^{\frac{1}{6}}\Pi(1-q^{2\nu})^2},$$

where q, as usual, equals $e^{i\pi\omega}$, and where ω denotes the ratio of the periods of the elliptic integrals, we get, in the three cases $n = 5, 7, 13$ respectively, $\tau = 125M^3$, $\tau = 49M^2$, $\tau = 13M$.

On the Theory of Groups. *By* Prof. Cayley, F.R.S.

[*Read May 9th*, 1878.]

I recapitulate the general theory so far as is necessary in order to render intelligible the quasi-geometrical representation of it which will be given.

Let $\alpha, \beta \ldots$ be functional symbols each operating upon one and the same number of letters, and producing as its result the same number of functions of these letters. For instance, $\alpha(x, y, z) = (X, Y, Z)$, where the capitals denote each of them a given function of (x, y, z).

Such symbols are susceptible of repetition and combination; $\alpha^2(x, y, z) = \alpha(X, Y, Z)$, or $\beta\alpha(x, y, z) = \beta(X, Y, Z)$, $=$ in each case three given functions of (x, y, z); and similarly for α^3, $\alpha^2\beta$, &c.

The symbols are not in general commutative, $\alpha\beta$ not $= \beta\alpha$; but they are associative, $\alpha\beta \,.\, \gamma = \alpha \,.\, \beta\gamma$, each $= \alpha\beta\gamma$, which has thus a determinate meaning.

Unity as a functional symbol denotes that the letters are unaltered, $1(x, y, z) = (x, y, z)$; whence $1\alpha = \alpha 1 = \alpha$.

The functional symbols *may* be substitutions; $\alpha(x, y, z) = (y, z, x)$, the same letters in a different order. Substitutions can be represented

by the notation $\begin{pmatrix} yzx \\ zxy \end{pmatrix}$, the substitution which changes xyz into yzx, or, as products of cyclical substitutions, $\alpha = \begin{pmatrix} yzx\,wvu \\ xyz\,uvw \end{pmatrix}$, $= (xyz)\,(uw)$, the product of the cyclical substitutions x into y, y into z, z into x, and u into w, w into u, the letter v being unaltered.

A set of symbols $\alpha, \beta, \gamma \ldots$, such that the product $\alpha\beta$ of each two of them (in each order, $\alpha\beta$ and $\beta\alpha$) is a symbol of the set, is a group. It is easily seen that 1 is a symbol of every group, and we may therefore give the definition in the form that a set of symbols $1, \alpha, \beta, \gamma \ldots$ satisfying the foregoing condition is a group. When the number of symbols (or terms) is $= n$, then the group is of the order n; and each symbol α is such that $\alpha^n = 1$, so that a group of the order n is in fact a group of symbolical n-th roots of unity.

A group is defined by means of the laws of combinations of its symbols. For the statement of these we may either (by the introduction of powers and products) diminish as much as may be the number of distinct functional symbols; or else, using distinct letters for the several terms of the group, employ a square diagram, as presently mentioned.

Thus, in the first mode, a group is $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ $(\alpha^2 = 1, \beta^3 = 1, \alpha\beta = \beta^2\alpha)$, where observe that these conditions imply also $\alpha\beta^2 = \beta\alpha$.

Or in the second mode, calling the symbols $(1, \alpha, \beta, \alpha\beta, \beta^2, \alpha\beta^2)$ of the same group $(1, \alpha, \beta, \gamma, \delta, \epsilon)$, or, if we please, (a, b, c, d, e, f), the laws of combination are given by one or other of the square diagrams:

	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	1	γ	β	ϵ	δ
β	β	ϵ	δ	α	1	γ
γ	γ	δ	ϵ	1	α	β
δ	δ	γ	1	ϵ	β	α
ϵ	ϵ	β	α	δ	γ	1

,

a	b	c	d	e	f
b	a	d	c	f	e
c	f	e	b	a	d
d	e	f	a	b	c
e	d	a	f	c	b
f	c	b	e	d	a

where, taking for greater symmetry the second form of the square, observe that the square is such that no letter occurs twice in the same line, or in the same column (or what is the same thing, each line and each column contains all the letters). But this is not sufficient in order that the square may represent a group; the square must be such that the substitutions by means of which its several lines are derived from any line thereof are (in a different order) the same substitutions

by which the lines are derived from a particular line, or say from the top line. These, in fact, are:

1

ab . cd . ef,
ace . bfd,
ad . be . cf,
aec . bdf,
af . bd . de,

where, for shortness, *ab*, *ace*, &c. are written instead of (*ab*), (*ace*), &c. to denote the cyclical substitutions *a* into *b*, *b* into *a*; and *a* into *c*, *c* into *e*, *e* into *a*, &c.; and it is at once seen that by the same substitutions the lines may be derived from any other line.

It will be noticed that in the foregoing substitution-group each substitution is *regular*, that is, composed of cyclical substitutions each of the same number of letters; and it is easy to see that this property is a general one; each substitution of the substitution-group must be regular.

By what precedes, the group of any order composed of the functional symbols is replaced by a substitution-group upon a set of letters the number of which is equal to the order of the group, and wherein all the substitutions are regular.

The general theory being thus explained, I endeavour to form a substitution-group with the twelve letters *abcdefghijkl*; and I assume that there is one substitution, such as *abc . def . ghi . jkl*, and another substitution, such as *agj . bfi . cek . dhl*. Observe that, if the twelve letters are to be thus arranged in two different ways as a set of four triads, without repetition of any duad, all the ways in which this can be done are essentially similar, and there is no loss of generality in taking the two sets of triads to be those just written down. But the substitution to be formed with either set of triads will be different according as any triad thereof, for instance *agj*, is written in this form or in the reversed form *ajg*. There are thus in all sixteen substitutions which can be formed with the first set of triads, and sixteen substitutions which can be formed with the second set of triads; and the relation of a triad of the first set to a triad of the second set is by no means independent of the selection of the triads out of the two sets respectively. To show this, take the two substitutions quite at random; suppose they are those written down above, say

$$\alpha = abc\,.\,def\,.\,ghi\,.\,jkl, \quad \beta = agj\,.\,bfi\,.\,cek\,.\,dhl;$$

and perform these in succession on the primitive arrangement $\Omega = abcdefghijkl$; the operation stands thus:

$$\beta\alpha\Omega = f\,e\,g\,k\,i\,h\,l\,b\,j\,c\,d\,a,$$
$$\alpha\Omega = b\,c\,a\,e\,f\,d\,h\,i\,g\,k\,l\,j,$$
$$\Omega = a\,b\,c\,d\,e\,f\,g\,h\,i\,j\,k\,l,$$

whence $$\beta\alpha, = afhbeijcgl\,.\,dk,$$

is not a regular substitution; and, by what precedes, α, β cannot belong to a group.

But take the substitutions to be α (as before) $= abc\,.\,def\,.\,ghi\,.\,jkl$, $\beta = ajg\,.\,bif\,.\,cek\,.\,dhl$, then we have

$$\beta\alpha\Omega = i\,e\,j\,k\,b\,h\,l\,f\,a\,c\,d\,g,$$
$$\alpha\Omega = b\,c\,a\,e\,f\,d\,h\,i\,g\,k\,l\,j,$$
$$\Omega = a\,b\,c\,d\,e\,f\,g\,h\,i\,j\,k\,l,$$

whence $$\beta\alpha = ai\,.\,be\,.\,cj\,.\,dk\,.\,fh\,.\,gl,$$

a regular substitution; and, for anything that appears to the contrary, α, β may belong to a group. It is convenient to mention at once that these two substitutions do, in fact, give rise to a group; viz., the square, diagram is

a	b	c	d	e	f	g	h	i	j	k	l
b	c	a	e	f	d	h	i	g	k	l	j
c	a	b	f	d	e	i	g	h	l	j	k
d	l	h	a	g	j	e	c	k	f	i	b
e	j	i	b	h	k	f	a	l	d	g	c
f	k	g	c	i	l	d	b	j	e	h	a
g	f	k	l	c	i	j	d	b	a	e	h
h	d	l	j	a	g	k	e	c	b	f	i
i	e	j	k	b	h	l	f	a	c	d	g
j	i	e	h	k	b	a	l	f	g	c	d
k	g	f	i	l	c	b	j	d	h	a	e
l	h	d	g	j	a	c	k	e	i	b	f

and the substitutions, obtained therefrom by writing successively each line over the top line, are

$$
\begin{array}{ll}
1 & = 1, \\
abc \,.\, def \,.\, ghi \,.\, jkl & \alpha, \\
acb \,.\, dfe \,.\, gih \,.\, jlk & \alpha^2, \\
ad \,.\, bl \,.\, ch \,.\, eg \,.\, fj \,.\, ik & \beta^2\alpha\beta^2, \\
aeh \,.\, bjd \,.\, cil \,.\, fkg & \beta\alpha^2, \\
afl \,.\, bkh \,.\, cgd \,.\, eij & \beta^2\alpha, \\
agj \,.\, bfi \,.\, cke \,.\, dlh & \beta^2, \\
ahe \,.\, bdj \,.\, cli \,.\, fgk & \beta\alpha^2\beta\alpha^2, \\
ai \,.\, be \,.\, cj \,.\, dk \,.\, fh \,.\, gl & \beta\alpha, \\
ajg \,.\, bif \,.\, cek \,.\, dhl & \beta, \\
ak \,.\, bg \,.\, cf \,.\, di \,.\, el \,.\, hj & \beta^2\alpha^2, \\
alf \,.\, bkh \,.\, cdg \,.\, eji & \beta^2\alpha\beta^2\alpha.
\end{array}
$$

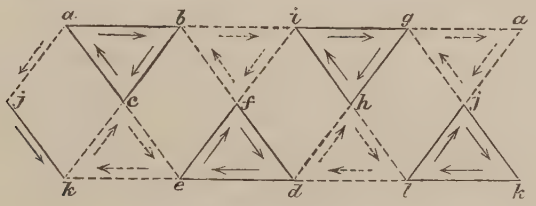
To explain the theory, I introduce the notion of a hemipolyhedron, or say a hemihedron, viz., this is a figure obtained from a polyhedron by the removal of certain faces. In a polyhedron each edge occurs twice (more properly it occurs in the two forms ab and ba), as belonging to two faces; but in a hemihedron one of these faces must always be removed, so that the edge may occur once only; and again (what is apparently, although not really, a different thing), we may remove two intersecting faces, leaving their edge of intersection; this edge is, in fact, then considered as a bilateral face $ab = ab \,.\, ba$, just as abc is a trilateral face $abc = ab \,.\, bc \,.\, ca$. Thus, if in a prism we remove the lateral faces, leaving the lateral edges, and leaving also the terminal faces, we have a hemihedron: thus, the prism being trilateral, say the faces of the hemihedron are abc, def, ad, be, cf, where ad, be, cf are the edges regarded as bilateral faces. And, for the present purpose, abc denotes the cyclical substitution a into b, b into c, c into a; and ad denotes in like manner the cyclical substitution (or interchange) a into d, d into a.

But the hemihedron about to be considered has no bilateral faces; it is, in fact, the figure composed of the 8 triangular faces of the octo-hexadron or figure obtained by truncating the summits of a hexahedron (or of an octahedron) so as to obtain a polyhedron of 8 triangular faces and 6 square faces, representing the faces of the octahedron and the hexahedron respectively. The faces of the octo-hexahedron may be taken to be

$$
\begin{array}{llllll}
abc, & def, & ghi, & jkl, & & \\
ajg, & bif, & cek, & dhl, & & \\
cbfe, & fihd, & hgjl, & jack, & agib, & klde,
\end{array}
$$

(where I observe in passing that the symbols are written in such manner that each edge ab occurs under the two opposite forms ab in abc and ba in $agib$). And then, omitting the square faces, represented by the third line, we have the hemihedron, wherein as before abc denotes the cyclical substitution a into b, b into c, c into a; and so for the other faces.

I represent this by a diagram, the lines of which were red and black, and they will be thus spoken of, but the black lines are in the woodcut continuous lines, and the red lines broken lines: each face indicates a



cyclical substitution, as shown by the arrows. The figure should be in the first instance drawn with the arrows, but without the letters, and these may then be affixed to the several points in a perfectly arbitrary manner; but I have in fact affixed them in such wise that the group given by the diagram, as presently appearing, may (instead of being any other equivalent group) be that group which contains the beforementioned substitution

$$\alpha = abc\,.\,def\,.\,ghi\,.\,jkl, \text{ and } \beta = ajg\,.\,bif\,.\,cek\,.\,dhl.$$

Observe that in the diagram, considering the lines to be drawn as shown by the arrows, there is *from* any given point whatever only one black line, and only one red line. Let B denote motion along a black line, R motion along a red line (always from a point to the next point); then B^2 will denote motion along two black lines successively, BR (any such symbol being read always from right to left) will denote motion first along a red line, and then along a black line, and so in other cases; a symbol or "route" ... $R^\delta B^\epsilon$ has thus a perfectly definite signification, determining the path when the initial point is given.

The diagram has the property that every route leading from any one letter to itself, leads also from every other letter to itself; or say a route leading from a to a, leads also from b to b, from c to c, ... from l to l; and we can thus in the diagram speak absolutely (that is, without restriction as to the initial point) of a route as leading from a point to itself, or say as being equal to unity; *it is in virtue of this property* that the diagram gives a group.

For, assuming the property, it at once follows (1) that two routes each leading say from the point a to the same point f, lead also from

any other point b to one and the same point g. Such routes are said to be equivalent, or equal to each other; and the number of distinct routes (including the route unity) is thus equal to the numbers of the letters, viz., we have only the routes from a to a, to b, ... to l, respectively; (2) a route leading from a point a to a point f leads from any other point b to a different point g; and (3) two routes leading from the same point a to different points b, c lead also from any other point f to different points k, l. Hence a given route leads from the several points $abc\ldots\ldots l$ successively to the same series of points taken in a different order, or we thus obtain a new arrangement of the points; and dealing in this manner successively with the routes from a to a, to b, ... to l, we obtain so many distinct arrangements, beginning with the letters $a, b, c \ldots\ldots l$ respectively, such that in no two of them does the same letter occupy the same place; we thus obtain a square of 12 such as that already written down, and which is, in fact, the same square, the several routes of course corresponding to the substitutions of the square. The hemihedron thus gives the foregoing group of 12.

Observe that the diagram is composed of the four black triangles representing the substitution $abc\,.\,def\,.\,ghi\,.\,jkl$, and of the four red triangles representing the substitution $ajg\,.\,bif\,.\,cek\,.\,dhl$; viz., these are independent substitutions which by their powers and products serve to express all the substitutions of the group; that they are sufficient appears by the diagram itself, in that every point thereof is (by black and red lines) connected with every other point thereof. The group might have contained three or more independent substitutions, and the diagram would then have contained the like number of differently coloured sets of lines. The essential characters are that the lines of any given colour shall form polygons of the same number of sides (but for different colours the polygons may have different numbers of sides; in particular, for any given colour or colours, the polygons may be bilaterals, represented each by a line with a double arrow pointing opposite ways); that there shall be *from* each point only one line of the same colour; that every point shall be connected with every other point; and finally, that every route leading from one point to itself shall lead also from every other point to itself. When these conditions are satisfied the foregoing investigation in fact shows that the diagram, or say the hemihedron, gives rise to a group.

It may be remarked that we can, if we please, introduce into the diagram a set of lines of a new colour to represent any dependent substitution of the group; thus, in the example considered, a substitution is $aeh\,.\,ljd\,.\,cil\,.\,fkg$, and if we draw these triangles in green (the arrows being from a to e, e to h, h to a, &c.), then there will be *from* each point one black line, one red line, and one green line; any route

... $G^{\gamma}R^{\beta}B^{\alpha}$ will thus be perfectly definite, and will have the same properties as a route composed of black and red lines only; and the theory thus subsists without alteration.

I remark, in conclusion, that the group of 12 considered above is, in fact, the group of 12 positive substitutions upon 4 letters *abcd*; viz., the substitutions are 1, *abc*, *acb*, *abd*, *adb*, *acd*, *adc*, *bcd*, *bdc*, *ab*.*cd*, *ac*.*bd*, *ad*.*bc*; the groups each contain unity, three substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3, and their identity can be easily verified.

On Conjugate Four-piece Linkages.

By A. B. Kempe, B.A.

[*Read May* 9, 1878.]

The plane linkage represented in Fig. 1, consisting of eight planes pivoted together [of which triangular portions only are shown in the figure], is in general a framework, *i.e.*, the various links cannot move about among themselves, but are fixed in relative position; and this is still so if any one piece be removed. In certain cases, however, this is not so, and the linkage is said to be "complete." It is the object of this paper to investigate these cases.

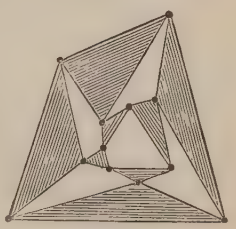


Fig. 1.

The investigation is important for this reason. The fundamental idea of linkages is constancy of distance; it is material therefore to investigate the various cases in which points on linkages are at a constant distance apart. Such points can be connected by links, and other links removed, so that we get new linkages having the same relations between certain of their parts as existed in the old ones. This constancy of distance does not prevail while the linkages go through the whole of their configurations,* as in "supplementable linkages," as these linkages may be termed in which points can be joined by new links, the algebraic relations between the

* The *form* of a linkage is the way in which the links are put together. The *dimensions* are the pivot distances. The *configuration* is the relative position of the links in a linkage of given form and dimensions.

parts break up. The factor-relations will be different in the case of the old and the new linkages, one or more relations being common to the two; and if the new link or links be added to the old linkage, and no link taken away, the common relations can alone exist,—we in fact eliminate "dead points." Now we know that of four-piece linkages the parallelogrammatic linkage alone is supplementable. Proceeding by steps, we come to six-piece linkages. The present paper is a contribution to the consideration of these six-piece linkages and in this way.

An examination of Fig. 1 will show that, if the linkage there shown is complete, twelve different six-piece supplementable linkages can be formed out of it by the removal of two adjacent links; the six-piece supplementable linkages being doubly supplementable by links which are themselves joinable.

The method I adopt for the purpose of investigating this eight-piece linkage is this:—It will be observed that the linkage can be divided in three different ways into two four-piece complete linkages such as that shown in Fig. 2. Let α, β, γ, δ be the pivots which were common to the two linkages when united, and join these by dotted lines, and let the resulting figure be called the "*connecting diagram.*" The relations between the lines and angles of this connecting diagram must be the same for the two linkages, and if we can find out for what linkages this is the case, we can find the eight-piece linkage. This is what is here done.

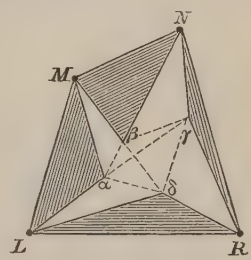


Fig. 2.

The investigation is, I venture to think, one of much interest on account of the novelty and symmetry of the results obtained. We shall, in the course of it, encounter both of Mr. Hart's five-bar parallel motions as special cases of more general results.

Before commencing, it will be convenient to define some terms which will be frequently employed. The two linkages which have a common connecting diagram will be termed "conjugate." It will also be convenient to call the linkage we treat of first in order the "first," and the next its "conjugate." The same parts of the conjugate will be affected with the same letters as the first, but they will be dashed. Those points on one connecting diagram which can be pivoted to points on the other connecting diagram are said to "correspond" to them. The figure LMNR, formed by joining the pivots of either linkage, is termed the "quadrilateral." The eight-piece linkage, formed by connecting the two four-piece linkages, I shall call the "compound" linkage.

In Fig. 3, let

$LM = a$, $\widehat{LaM} = a$,

$La = a_1$,

$Ma = a_2$, area of $LaM = A$;

and let corresponding letters stand for the same parts of the other pieces in which

$MN = b$, $NR = c$, $RL = d$.

Also let both angles at $L = L$,

" " $M = M$,

" " $N = N$,

" " $R = R$,

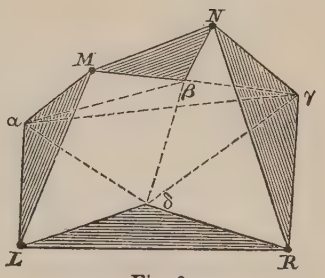


Fig. 3.

this equality of angles at L, M, N, and R being the only restriction upon the otherwise perfect generality of the linkage.

Let $\delta a = \lambda$,

$a\beta = \mu$, $MR = V$,

$\beta\gamma = \nu$,

$\gamma\delta = \rho$, $LN = W$.

We can readily find the following,

$$\alpha + \gamma = \beta + \delta \quad \text{.....(1)},$$

$$\frac{a^2}{A} + \frac{c^2}{C} = \frac{b^2}{B} + \frac{d^2}{D} \quad \text{.....(2)},$$

$$\frac{a^2 c^2}{AC \sin\alpha \sin\gamma} = \frac{b^2 d^2}{BD \sin\beta \sin\delta} = P \quad \text{.....(3)},$$

$$\left[\frac{a^2}{A}\cot\alpha + \frac{c^2}{C}\cot\gamma\right] - \left[\frac{b^2}{B}\cot\beta + \frac{d^2}{D}\cot\delta\right]$$

$$= -\frac{P}{2}\sin(\beta-\alpha)\sin(\gamma-\beta) \quad \text{.....(4)},$$

$$\frac{a_2}{b_1} = \frac{bA}{aB} \quad \text{and three other similar equations.....(5)},$$

$$a_1 a_2 = \frac{2A}{\sin\alpha} \quad \text{" " .....(6)},$$

$$\cos\alpha = \frac{a_1^2 + a_2^2 - a^2}{2a_1 a_2} \quad \text{" " .....(7)},$$

$$\sin^2 L = \frac{\frac{D}{d^2} - \frac{A}{a^2}}{\cot\delta - \cot\alpha} \quad \text{" " .....(8)},$$

and, from (5), $a_1 b_1 c_1 d_1 = a_2 b_2 c_2 d_2$ .....(9).

We have also

$$\frac{\lambda^2 \sin\alpha \sin\delta}{2\sin(\alpha-\delta)}\left[\frac{a^2}{A}-\frac{d^2}{D}\right] = V^2-\left[\frac{a^2}{A}-\frac{d^2}{D}\right](A-D) \quad \text{.....(10)},$$

and similar equations giving μ, ν, and ρ. Also the angle between μ and ρ is

$$\widehat{MNL}-\widehat{NLR}-\widehat{M\beta\alpha}+\widehat{R\delta\gamma}+(M-R)$$
$$= MNL-\widehat{NLR}-\widehat{M\beta\alpha}+\widehat{R\delta\gamma}+(\gamma-\beta),$$

and we have a similar expression for the angle between λ and ν...(11).

Case I.

Let $\alpha=\beta=\gamma=\delta$ and therefore the four triangles be similar. The connecting diagram becomes a parallelogram, and if the conjugate linkage be a duplicate of the first, we have a correspondence between

α and γ',
β „ δ',
γ „ α'
δ „ β',

and we thus get a complete eight-piece linkage, which is shown in Fig. 4.

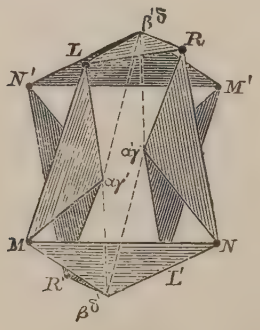


Fig. 4.

The other pairs of conjugate linkages into which the compound linkage can be divided are related to each other in precisely the same way as the first pair, their connecting diagrams being parallelograms.

CASE II.

Let $$A+C=B+D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

so that $$\left.\begin{aligned} \lambda^2 \sin\alpha \sin\delta &= \nu^2 \sin\beta \sin\gamma \\ \mu^2 \sin\alpha \sin\beta &= \rho^2 \sin\delta \sin\gamma \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Then, if a second linkage be formed in which

$$\left.\begin{aligned} a' &= ka, & A' &= A, & \alpha' &= \pi-\gamma \\ b' &= kb, & B' &= B, & \beta' &= \pi-\delta \\ c' &= kc, & C' &= C, & \gamma' &= \pi-\alpha \\ d' &= kd, & D' &= D, & \delta' &= \pi-\beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (14),$$

which, it will be seen by equations (1) to (8), can always be done if, (4),

$$\left[\frac{a^2}{A}\cot\gamma+\frac{c^2}{C}\cot\alpha\right]-\left[\frac{b^2}{B}\cot\delta+\frac{d^2}{D}\cot\beta\right] = -\frac{P}{2}k^2\sin(\beta-\alpha)\sin(\gamma-\beta),$$

i. e., $$k^2-1=\frac{\frac{A}{a^2}+\frac{C}{c^2}}{\sin\beta\sin\delta}\sin(\beta+\delta)=\frac{\frac{B}{b^2}+\frac{D}{d^2}}{\sin\alpha\sin\gamma}\sin(\alpha+\gamma) \quad \ldots\ldots (15).$$

Then we have, by (13),

$$\left.\begin{aligned} \lambda' &= \nu \\ \mu' &= \rho \\ \nu' &= \lambda \\ \rho' &= \mu \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16);$$

and, by (11), (1), and (5), the angles between λ' and ν', μ' and ρ' are the same as those between ν and λ, ρ and μ.

Thus the connecting diagram of the linkage $[abcd]$ is the same as that of $[a'b'c'd']$, there being a correspondence between

α and γ',
β and δ',
γ and α',
δ and β'.

The quadrilaterals of the two linkages are similar, but are in the compound linkage laid face to face.

The other pairs of conjugate linkages into which the compound linkage is divisible have the same relations to each other as the linkages $[abcd]$ and $[a'b'c'd']$. It is worthy of notice that the quadrilaterals of all the conjugate linkages have only six variable angles.

We now proceed to some special cases of this Case II.

Case II. (*a*).

This is the case in which the linkage of Case II. becomes a bar linkage.

In this case,
$$\left.\begin{aligned} a &= a_1 + a_2 \\ b &= b_1 + b_2 \\ c &= c_1 + c_2 \\ d &= d_1 + d_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

and we have similar expressions for a', &c.

The areas A, B, C, D vanish, and $\alpha, \beta, \gamma, \delta$ each equal π.

The equation (12), being put into the form

$$\frac{ABC}{BCD} - \frac{BC}{CD} + \frac{C}{D} - 1 = 0,$$

becomes
$$\frac{aa_2b_2c_2}{db_1c_1d_1} - \frac{bb_2c_2}{dc_1d_1} + \frac{cc_2}{dd_1} - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (18),$$

and we have a similar equation in a', &c.

We have also, from (5),

$$\frac{a_2}{b_1} = \frac{a_2'}{b_1'}, \text{ and similar equations } \ldots\ldots\ldots\ldots\ldots\ldots (19);$$

and from (6), recollecting that $\alpha' = \pi - \gamma$, &c.,

$$a_1a_2c_1c_2 = a_1'a_2'c_1'c_2', \quad b_1b_2d_1d_2 = b_1'b_2'd_1'd_2' \quad \ldots\ldots\ldots\ldots\ldots (20).$$

By the aid of (5), (6), and (7), we can put (15) into the form

$$k^2 = \frac{abcd - a_1b_1cd - b_2c_2ad - c_1d_1ba - d_2a_2bc}{abcd},$$

which can be further simplified into

$$k^2 = -\frac{(a_1c_1 - a_2c_2)(b_1d_1 - b_2d_2)}{abcd} \quad \ldots\ldots\ldots\ldots\ldots (21).$$

Case II. (*b*).

Let $k = 0$.

Then the sides of $[a'b'c'd']$ vanish, and therefore the areas A, B, C, D vanish, and the linkage becomes a bar one.

Hence, by (21), and (9),
$$\left.\begin{aligned} a_1c_1 &= a_2c_2 \\ b_1d_1 &= b_2d_2 \end{aligned}\right\}.$$

Let
$$a_1 = ea, \quad b_1 = e'b.$$

Then (18) becomes

$$\frac{a^2}{e'} + \frac{c^2}{1-e'} = \frac{b^2}{1-e} + \frac{d^2}{e} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

and (19) becomes

$$\frac{(1-e)}{e'} \cdot \frac{a}{b} = -\frac{a'_1}{b'_1},$$

$$\frac{(1-e')}{(1-e)} \cdot \frac{b}{c} = -\frac{b'_1}{c'_1},$$

$$\frac{e}{(1-e')} \cdot \frac{c}{d} = -\frac{c'_1}{d'_1},$$

$$\frac{e'}{e} \cdot \frac{d}{a} = -\frac{d'_1}{a'_1};$$

whence

$$\frac{a'^2_1}{c'^2_1} = \frac{a^2}{c^2}\left[\frac{1-e'}{e'}\right]^2,$$

$$\frac{b'^2_1}{d'^2_1} = \frac{b^2}{d^2}\left[\frac{e}{1-e}\right]^2,$$

and (20) becomes

$$a'^2_1 c'^2_1 = a^2c^2e'^2(1-e)^2,$$

$$b'^2_1 d'^2_1 = b^2d^2e'^2(1-e')^2.$$

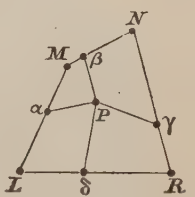


Fig. 5.

Thus,

$$a'^4 = a^4\frac{e^2}{e'^2}(1-e)^2(1-e')^2,$$

and we can get corresponding values for b', c', d'.

The form of the compound linkage is shown in Fig. 5. Fixing LR, and removing the links $P\beta$ and $P\delta$, we have a circle-describing apparatus. If we transpose the bars αP and γP, we get a second such apparatus, the two forms corresponding to the two values of e' given by (22). It is to be noticed that this removal of the links $P\beta$ and $P\delta$, which are jointed to opposite sides of the quadrilateral $LMNR$, cannot be effected in the general form of the compound linkage given in Fig. 1.

One special case of this form is of particular interest.

Case II. (b').

Let $e' = \infty$.

Then, by (22),

$$e = \frac{d^2}{d^2-b^2},$$

and

$$a'^4_1 = a^4e^2(1-e^2);$$

i. e.,

$$a'_1 = \frac{adb}{d^2-b^2}, \quad c'_1 = \frac{cbd}{d^2-b^2};$$

and d'_1 and b'_1 are infinite.

The apparatus becomes Mr. Hart's second parallel motion, see Vol. VIII., p. 288.

Case II. (*c*).

The result in the last section can also be arrived at in another way which is perhaps more elegant. Take as our four-bar linkage that shown in Fig. 6, in which

$$c = \infty, \quad d = c + h,$$

where h is finite.

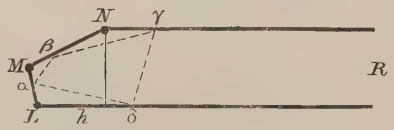


Fig. 6.

The point R being at an infinite distance off, the bars c and d will be parallel, and will approach and recede from each other directly.

We have d_2 and c_1 finite, d_1 and c_2 infinite; and

$$d_1 = d - d_2 = c + h - d_2, \quad \text{and} \quad c_2 = c - c_1.$$

Hence (18) takes the form

$$c_1 = \frac{b_2}{h}\left[\frac{aa_2}{b_1} - b\right],$$

and (9) becomes $$a_1 b_1 c_1 = a_2 b_2 d_2;$$

whence $$d_2 = \frac{a_1}{h}\left[a - \frac{bb_1}{a_2}\right];$$

also (21) becomes $$k^2 = \frac{a_2 b_1}{ab},$$

and we have similar equations for $a'b'c'd'$, &c.

The equations (20) become

$$a_1 a_2 c_1 = a_1' a_2' c_1' k,$$
$$b_1 b_2 d_2 = b_1' b_2' d_2' k;$$

and we also have (19).

The resulting compound linkage has two pair of bars which are parallel and move directly to and from each other.

Now let $c_1 = 0$; then $b_2 = 0$, and β, γ, N all coincide. The compound linkage is shown in Fig. 7.

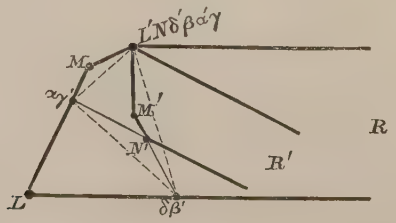


Fig. 7.

The bars NR and $L'R'$ become superfluous, and may be removed, and Mr. Hart's linkage remains. The connecting diagram is in this case a triangle.

CASE II. (d).

Let $k = 1$.

Here, by (15), we have either

$$\sin(\alpha+\gamma) = 0,$$

or

$$\frac{B}{b^2} + \frac{D}{d^2} = 0.$$

I. Let $\sin(\alpha+\gamma) = 0$;

$$\therefore \quad \alpha = n\pi - \gamma;$$

$\therefore$ by (1), $\beta = n\pi - \delta.$

Hence, by (13), $\lambda^2 = \nu^2,$

$$\mu^2 = \rho^2.$$

The connecting diagram must therefore be a contra-parallelogram. It cannot be a parallelogram, because that would involve all the triangles on a, b, c, d being similar, as in Case I., which would be inconsistent with $\sin(\alpha+\gamma) = 0.$

A bar form of one conjugate is shown in Fig. 8; the other is a duplicate.

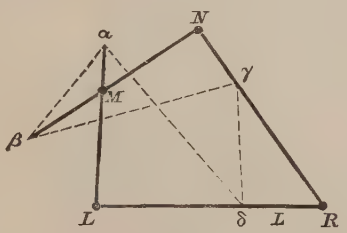


Fig. 8.

There is a special form of the bar case which is of interest.

Let $\alpha\beta$ be always parallel to $\gamma\delta$. For this we must have (Fig. 3)

$$M = R, \quad i.e., \quad M+N = N+R;$$

$$\therefore \quad \beta = \gamma, \quad \text{and} \quad \therefore \quad \alpha = \delta.$$

But $$\sin^2 L = \frac{\frac{D}{d^2} - \frac{A}{a^2}}{\cot\delta - \cot a} = \infty, \quad \text{unless} \quad \frac{D}{d^2} = \frac{A}{a^2};$$

so $$\sin^2 N = \frac{\frac{B}{b^2} - \frac{C}{c^2}}{\cot\beta - \cot\gamma} = \infty, \quad \text{unless} \quad \frac{B}{b^2} = \frac{C}{c^2}.$$

We must therefore, by (12), either have

$$a^2+c^2=b^2+d^2,$$

or $$A=0,\quad B=0,\quad C=0,\quad D=0.$$

The former case involves the linkage, being one with perpendicular diagonals, and the triangles on the sides of the quadrilateral being all similar, and therefore the connecting diagram a parallelogram, which is, as before pointed out, inconsistent with

$$\sin(\alpha+\gamma)=0.$$

In the latter case the linkage becomes a bar one. It is shown in Fig. 9.

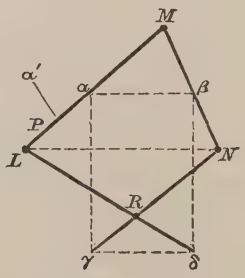


Fig. 9.

In it we have

$$a_1=(1-s)\,a,$$
$$b_2=(1-s)\,b,$$
$$c_1=(1+s)\,c,$$
$$d_2=(1+s)\,d,$$

where $$s=\frac{[a^2-b^2]+[c^2-d^2]}{[a^2-b^2]-[c^2-d^2]}.$$

The four points α, β, δ, γ lie at the angles of a rectangle, whatever be the configuration of the linkage. It is noticeable that, if we merely desire to make the linkage without wishing for any special values of a, b, c, d, we can draw any rectangle $\alpha\beta\delta\gamma$, and any diameter LN parallel to $\alpha\beta$, and then complete the figure. The resulting linkage will be such that $\alpha\beta\delta\gamma$ is always a rectangle. The lengths a, b, c, d are, of course, entirely at our disposition if we desire it, the points α, β, γ, δ being determined by the value of s as given. It is somewhat remarkable that it should be possible, in a linkage of which the pivot-distances are arbitrary, to obtain four points making a figure with constant angles.

We may from this bar-linkage obtain, by a very simple geometrical construction, a piece-linkage, which falls under this case, and in which the connecting diagram has this property of angular constancy.

In Fig. 10 the figure $LMNR$ is the same as in Fig. 9.

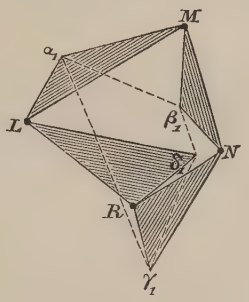


Fig. 10.

Bisect La in P, $N\beta$ in Q, $N\gamma$ in S, $L\delta$ in T (see Fig. 9).

Make
$$Pa_1 = Pa, \quad \text{and} \quad \widehat{aPa_1} = \epsilon;$$
$$Q\beta_1 = Q\beta, \quad \text{and} \quad \widehat{\beta Q\beta_1} = \epsilon;$$
$$S\gamma_1 = S\gamma, \quad \text{and} \quad \widehat{\gamma S\gamma_1} = \epsilon;$$
$$T\delta_1 = T\delta, \quad \text{and} \quad \widehat{\delta T\delta_1} = \epsilon.$$

Then $\alpha_1\gamma_1$ is parallel to $\beta_1\delta_1$, and $\alpha_1\beta_1 = \gamma_1\delta_1$, and the angles of α_1, β_1, δ_1, γ_1 are constant, the figure being symmetrical.

The other correspondences which are suggested by the symmetry of the connecting diagrams in Figs. 9 and 10, fall under Case III.

II. Let
$$\frac{B}{b^2} + \frac{D}{d^2} = 0,$$
$$\therefore \quad \frac{A}{a^2} + \frac{C}{c^2} = 0.$$

Thus
$$A = \frac{\sigma a^2}{a^2 - c^2},$$
$$B = \frac{\sigma b^2}{b^2 - d^2},$$
$$C = \frac{\sigma c^2}{c^2 - a^2},$$
$$D = \frac{\sigma d^2}{d^2 - b^2}.$$

In this case, the quadrilaterals of the two conjugates being the same, their connecting diagrams may be exhibited on one of them. This

was of course also the case in the example just treated of, but there the two connecting diagrams coincided; here it is not so.

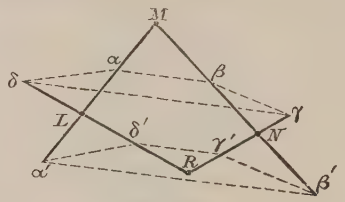


Fig. 11.

In Fig. 11, a bar case, the two connecting diagrams are shown. This case does not present any points of special interest.

Case III.

Let (12) be satisfied, and let

$$M = R,$$

$$\therefore \quad M+N = N+R,$$

$$\therefore \quad \beta = \gamma,$$

and $$\therefore \quad \alpha = \delta.$$

Then $$\mu = \rho, \text{ by (13)},$$

and the triangles B and C are similar, as also the triangles A and D.

Also the angle between μ and ρ is

$$N-L = \overline{N+M} - \overline{M+L} = \alpha-\beta.$$

Thus, if we turn all the triangles of Fig. 3 over to opposite sides of a, b, c, and d, the figure $\alpha\beta\gamma\delta$ remains the same in dimensions, and its form is merely changed in the manner shown in Figs. 12 and 13.

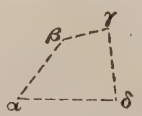


Fig. 12.

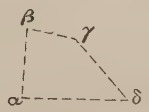


Fig. 13.

If therefore we construct the two linkages which give these two connecting diagrams, and dash the letters in one, we can make a correspondence between

α and δ',
δ and α',
β and γ',
γ and β',

by turning them face to face. The same thing may be done with the triangles in Case I.

The resulting compound linkage may be dissected into two quadru-

planes, and we may obtain the compound linkage by connecting any two quadruplanes, in which the area and angles of the connecting diagrams are the same. The quadruplane is so well known that I shall

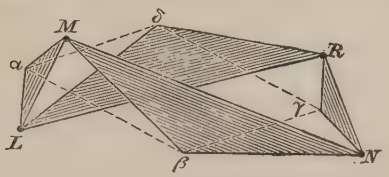


Fig. 14.

merely give a figure of it, Fig. 14, and point out that

$$LR = MN,$$
$$ML = RN,$$

the triangles are all similar,

$$\alpha\delta \,.\, \delta\gamma = D - A = [-B] - [-C],$$

and the angles of the connecting diagram are all constant.

In the particular case in which the sides of one quadruplane become infinite, we get a double Sylvester-Kempe parallel motion, of which the bar case was discovered by Mr. Hart and is known by his name.

In the bar case the connecting diagram in Fig. 12 takes the symmetrical form shown in Fig. 15.

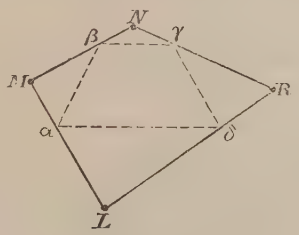


Fig. 15.

Figure 9 is a special case of this.

The special case in which β and γ move up and coincide with N, the connecting diagram becoming an isosceles triangle, is shown in Fig. 16, which exhibits the compound linkage resulting from joining the two

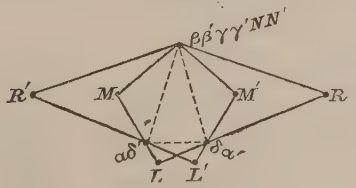


Fig. 16.

conjugates. Any two of the bars meeting at N may be removed.

Case IV.

The special case requires notice, in which

$$a^2+c^2 = b^2+d^2,$$

i.e., in which the diagonals of the quadrilateral of the four-piece linkage are at right angles to each other.

In this case, if we make the four triangles of Fig. 3 similar, they satisfy the equation

$$A+C = B+D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

the connecting diagram being a parallelogram. The *sides* of the triangles of the conjugate linkage of Case III. are the same as those of the first, the vertical angles being, as required by (14), the supplements of those of the first. Consequently, if we take the conjugates of Case I. and Case II., and connect them together,—which we can do, as the connecting diagram is the same in both cases,—we get, by combining them, a compound linkage which will dissect into conjugate "Kites."

The bar form of this compound linkage is shown in Fig. 17.

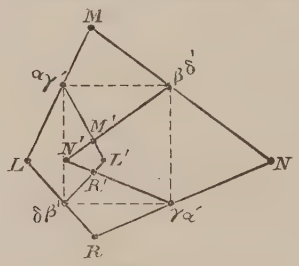


Fig. 17.

It is noticeable that, if the vertical angles of the similar triangles on the four sides of the quadrilateral in Case IV. are right angles, the connecting diagram becomes a straight line.

Case V.

In the cases hitherto discussed, the angles at a pivot have all been equal or supplements. In the present case there is no such restriction. In Fig. 18 any quadrilateral figure $\alpha\beta\gamma\delta$ is taken as the connecting diagram, and similar triangles $\alpha M\beta$, $\beta N\gamma$, $\gamma R\delta$, $\delta L\alpha$, are described on the sides of it.

The vertices L, M, N, R then form a parallelogram. If then the parallelogrammatic linkage be constructed as in the figure, the connecting diagram will always remain

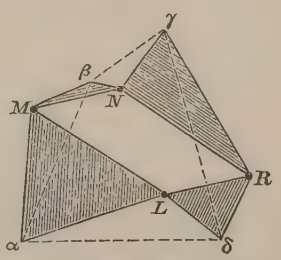


Fig. 18.

similar to itself. Hence, if we form another linkage upon the same diagram in the same manner, the two will be conjugate. The other quadrilaterals of the compound linkage are all similar, and may be of any dimensions.

Case VI.

This case does not belong to the same class as those hitherto treated of, as the compound linkage can only be dissected into two pairs of conjugate linkages. As, however, it is directly suggested by Case V., I have thought it better to include it here.

It is clear, from Fig. 18, that not only does the figure $\alpha\beta\gamma\delta$ remain similar to itself, but, if we join $\alpha\gamma$ and $\beta\delta$, the figure $\alpha\gamma\beta\delta$ does so also. Hence, treating this figure as we did $\alpha\beta\gamma\delta$, we get a conjugate linkage, which, if connected to the first, gives us an eight-piece compound linkage. The correspondence is not, as in former cases, between adjacent points and adjacent points, but is between

α and δ',
β and β',
γ and γ',
δ and α'.

The resulting linkage is only divisible into four-piece linkages in one other way.

The compound linkage is shown in Fig. 19.

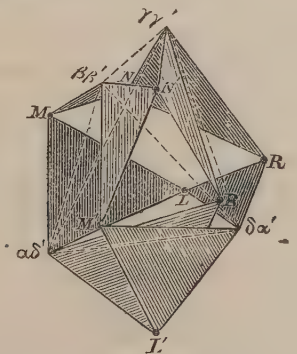


Fig. 19.

I have not as yet found any other linkages of this description.

June 13*th*, 1878.

Prof. H. J. S. SMITH, F.R.S., Vice-President, in the Chair.

Mr. J. D. H. Dickson, M.A., Fellow and Tutor of St. Peter's College, Cambridge, was proposed for election, and Mr. T. R. Terry was admitted into the Society.

Dr. Hirst, F.R.S., gave an account of a paper by M. Halphen, on "the Characteristics of Systems of Conics."

Mr. J. J. Walker read a paper on "a method in the Analysis of Plane Curves."

Mr. Tucker communicated papers by Mr. McColl, on "the Calculus of Equivalent Statements, II."; by Mr. C. J. Monro, on "the Flexure of Spaces"; by Mr. S. Roberts, F.R.S., on "the Decomposition of certain Numbers into sums of two Square Integers by continued Fractions"; by Mr. R. Rawson, on "a new method of finding Differential Resolvents of Algebraical Equations."

Questions were asked by Prof. Cayley, F.R.S. (Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?); by Mr. Merrifield, F.R.S., on "the uniform Distribution of Points in Space"; by Mr. Tucker, in connection with the announcement made in "Nature" (Vol. xvii., p. 104), of a second exception to Fermat's statement that all numbers of the form $2^2 + 1$ are primes.*

The following presents were made to the Society, some at the Meeting and some in the vacation :—

"Educational Times," June, July, 1878.

"Ueber Gleichungen siebenten Grades," zweite Mittheilung von F. Klein, in München.

"Proceedings of the Royal Society," Vol. xxvii., No. 187.

"Monatsbericht," März, April, Mai, 1878.

"Bulletin de la Société Mathématique de France," Tome vi., No. 4, 1878.

"Bulletin des Sciences Mathématiques," 2^{me} Série, Tome ii., Janvier 1878.

"Physical Society of London, Proceedings," Vol. ii., Pt. iv., Jan. to June, 1878.

"Crelle's Journal," 85 Band, 2es Heft; Berlin, 1878.

Rayleigh's (Lord) "Theory of Sound," Vol. ii.; from the Author.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. iii., Pt. 2; New Haven, 1878.

"Atti della R. Accad. dei Lincei," anno cclxxv., 1877–8, Serie terza; "Transunti," Vol. ii., Fasc. 6°, Maggio 1878.

* A third case of exception, discovered by the same Russian clergyman, has subsequently been given in "Nature" (No. 456, July 25th).

"American Journal of Mathematics, Pure and Applied" (published under the auspices of the Johns Hopkins University), Vol. i., Nos. 1, 2; Baltimore, 1878.

Sur la Théorie des Caractéristiques pour les Coniques.

Par M. HALPHEN.

[*Read June* 13*th*, 1878.]

Dans deux communications adressées à l'Académie des Sciences de Paris,* j'ai indiqué des modifications qu'il est nécessaire de faire subir à la théorie connue des caractéristiques pour les coniques. Je me propose de donner ici un aperçu des résultats que j'ai obtenus dans l'étude de cette théorie, et que j'ai développés dans un mémoire qui n'a pu être publié jusqu'à présent.

I. *Position de la question.*

1. On donne, je le rappelle, le nom de *système* de coniques dans un plan à l'ensemble des coniques de ce plan qui satisfont à quatre conditions communes. Le problème principal auquel se rapporte la théorie des caractéristiques est de trouver le nombre des coniques, appartenant à un système donné, qui satisfont, d'autre part, à une condition également donnée. En conséquence, cette théorie doit enseigner quels sont les éléments numériques, ou *caractéristiques*, qu'il faut connaître pour le système et pour la condition, et comment on doit combiner ces éléments numériques afin d'obtenir le nombre cherché. Pour qu'il y ait lieu à une théorie sur ce sujet, il faut évidemment supposer entre ces deux données, le système et la condition, une certaine *indépendance;* et on n'a jamais manqué de le faire. Mais cette indépendance doit être définie avec précision, et c'est par là que j'entrerai en matière.

2. Disons tout d'abord que la question est portée sur le terrain de la géométrie projective; par suite, toute condition imposée à une conique devra être mise sous forme projective. Pour la mettre sous une telle forme, on peut convenir d'employer un procédé unique, comme je vais l'expliquer.

Soient, dans un plan, quatre points fixes t, u, v, w et un autre point x, et désignons par t, t', ... X, X' les coordonnées de ces points par rapport à deux axes. Pour abréger, employons la notation

$$(tuv) = \begin{vmatrix} t & t' & 1 \\ u & u' & 1 \\ v & v' & 1 \end{vmatrix},$$

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* 4 Septembre et 13 Novembre 1876. V. Comptes Rendus, T. lxxxiii., pp. 537 et 886.

et posons :

$$(1)\ldots\ldots\ldots\ldots\ldots x = \frac{(vwX)\,(uvt)}{(uvX)\,(vwt)}, \quad x' = \frac{(wuX)\,(uvt)}{(uvX)\,(wut)}.$$

Si les quatre points fixes occupaient des positions particulières, savoir u à l'infini sur l'axe des X, v à l'infini sur l'axe des X', w à l'origine des coordonnées, et que t eût ses deux coordonnées égales à l'unité, il résulterait de cette hypothèse : $x = X$, $x' = X'$. D'autre part, x et x' sont les rapports anharmoniques de deux faisceaux de droites obtenus en joignant soit v, soit u, aux quatre autres points. Si donc on fait le changement de coordonnées marqué par les équations (1), x, x' étant les anciennes coordonnées et X, X' les nouvelles, on exécute ainsi une transformation homographique quelconque en introduisant, à cet effet, quatre points t, u, v, w arbitraires, mais fixes. Donc *une condition quelconque imposée à une figure peut être traduite, sous forme projective, par une relation numérique entre les coordonnées de cette figure et celles de quatre points.*

3. Revenons maintenant aux coniques, et posons la question :

Soit Φ *une condition projective donnée entre une conique et quatre points* t, u, v, w ;

Soit, d'autre part, S *un système de coniques donné ;*

On envisage les coniques du système S *qui satisfont à la condition* Φ, *et l'on demande de discuter le problème de la recherche de ces coniques, en examinant :*

1°. *S'il y a des solutions indépendantes des points* t, u, v, w, *et quelles sont ces solutions ;*

2°. *Quel est le nombre des solutions distinctes qui dépendent de ces points, lorsqu'ils restent indéterminés.*

La discussion complète exigerait, en troisième lieu, l'examen des diverses particularités que peuvent présenter ces dernières solutions, en se réunissant soit entre elles, soit avec les précédentes, lorsque les points arbitraires de Φ occupent des positions particulières. Cette dernière partie de la discussion est formellement exclue de notre programme, et c'est en quoi consiste *l'indépendance* supposée entre le système S et la condition Φ.

4. A l'égard de la première catégorie de solutions, on peut dès l'abord être fixé sur leur nature. Par une transformation homographique, une conique a peut être transformée en une autre conique arbitraire a'. Soit a une conique, qui, dans un système S, satisfasse à une condition Φ. Par une transformation homographique, changeons a en a' et, en meme temps, les points t, u, v, w de Φ en t', u', v', w'. La conique a' satisfait à la condition Φ ainsi transformée. Si maintenant on suppose que a satisfasse à Φ, quels que soient t, u, v, w, il en est de même de a'. Donc une conique arbitraire a' satisfait à la condition Φ, ce qui est

impossible. Donc a ne peut être une conique. Donc *les solutions qui sont indépendantes des arbitraires de la condition* Φ *ne sont pas des coniques, mais des figures particulières, limites de coniques dans le système, ou coniques* dégénérées.

En second lieu, si S est un système de véritables coniques, les solutions qui dépendent des arbitraires de Φ, et qui, en conséquence, varient dans le système avec ces arbitraires, sont de véritables coniques. Ainsi la distinction ci-dessus des solutions en deux catégories, revient exactement à la distinction des solutions étrangères fournies par les coniques dégénérées, et des solutions fournies par de véritables coniques. C'est de ces dernières qu'il s'agit de déterminer le nombre. Je vais exposer ici la solution du problème dans un cas particulier, pour lequel les résultats se démontrent avec une extrême facilité, et ont exactement la même forme que dans le cas le plus général. En ce qui concerne le cas général, je renverrai, pour la démonstration, au mémoire dont j'ai parlé plus haut, et qui sera publié ultérieurement.

II. *Systèmes de coniques, et coniques dégénérées.*

5. Considérons d'abord, pour plus de simplicité, des coniques conjuguées par rapport à un triangle fixe. Soient $Q=0$, $R=0$, $S=0$ les équations des côtés de ce triangle, dont je désignerai par q, r, s les sommets respectivement opposés à ces côtés. L'équation d'une quelconque de ces coniques est:

$$(2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad g_1Q^2+g_2R^2+g_3S^2=0.$$

Cette conique dégénère en deux droites ou en une droite suivant que un ou deux des coefficients g deviennent nuls. Mais, si l'on envisage un système de coniques (2), cette observation ne suffit pas. Les g sont alors des fonctions d'une variable ω, et il est nécessaire d'examiner quelles sont les diverses manières dont ces fonctions parviennent à la limite zéro. De là trois modes de dégénérescence á distinguer:

1°. *Dégénérescence* A.—Un seul coefficient devient nul, par exemple g_1. La conique se réduit à deux droites passant en q. Il est aisé de voir que, pour g_1 infiniment petit, les tangentes menées à la conique par un point arbitraire la touchent en deux points infiniment voisins de q. Ainsi ce mode de dégénérescence est caractérisé par ce fait que *les tangentes menées d'un point quelconque à la conique font entre elles un angle infiniment petit.*

2°. *Dégénérescence* A′.—Deux coefficients infiniment petits, mais *d'un même ordre;* par exemple g_1 et g_2. La conique-limite est la droite S. Ce mode de dégénérescence est caractérisé par ce fait que *la conique est rencontrée par une droite quelconque en deux points infiniment voisins;* mais les tangentes issues d'un point quelconque ne coïncident pas.

3°. *Dégénérescence* B. — Deux coefficients infiniment petits, mais d'ordres différents ; par exemple g_2 du 1er ordre, et g_1 d'ordre supérieur, $1+h$. Les deux caractères précédents existent alors simultanément. *Une droite quelconque coupe la conique en deux points dont la distance d est infiniment petite. Les tangentes issues d'un point quelconque font entre elles un angle δ infiniment petit. Il y a alors lieu de considérer un nombre positif h, tel que $d^h : \delta$ ait une limite finie, différente de zéro.*

On remarquera que, d'après les propriétés des fonctions algébriques, il existe toujours un tel nombre h commensurable, si, comme nous le supposerons, le système est algébrique.

6. En même temps que le système (a), contenant la conique dégénérée a, considérons un système corrélatif (a'), et soit a' la conique qui y correspond à a. Soient, pour a', désignés par d', δ', h' les éléments analogues. Il est visible que d' est du même ordre infinitésimal que δ, et δ' du même ordre que d. Donc h' est l'inverse de h. Donc : *dans deux systèmes corrélatifs, deux coniques dégénérées suivant le mode* B *se correspondent, et leurs nombres caractéristiques h sont réciproques. Les coniques dégénérées suivant le mode* A *correspondent à des coniques dégénérées suivant le mode* A′.

Ces résultats sont d'ailleurs mis en évidence au moyen de l'équation :

$$g_2 g_3 Q^2 + g_3 g_2 R^2 + g_1 g_2 S^2 = 0,$$

qui représente la polaire réciproque de (2) par rapport à une conique fixe.

7. Cette étude, faite pour un système de coniques conjuguées par rapport à un triangle fixe, s'applique à un système quelconque, en vertu d'une remarque très-simple.

Soit, dans un système (a), une conique a qui dégénère. Je prends arbitrairement un point s et une droite Q passant en s. Les points y, z, où a coupe Q, peuvent être différents ou infiniment voisins ; mais, en tous cas, leurs limites diffèrent de s, qui est arbitraire sur Q. Donc le conjugué harmonique de s par rapport à y, z, a une limite différente de s. On peut répéter le même raisonnement pour toute droite Q menée par s, et conclure que la polaire de s a une limite qui ne passe par s. De même aussi le pôle de Q a une limite qui n'est pas sur Q. Donc, *dans la série des triangles respectivement conjugués par rapport à chacune des coniques d'un système, et ayant un sommet fixe ainsi qu'un des côtés aboutissant à ce sommet, le triangle qui répond à une conique dégénérée est un triangle véritable, c'est-à-dire non dégénéré.* On peut donc encore raisonner sur l'équation (2) comme ci-dessus, et en tirer les mêmes conclusions, que ne modifie en rien la mobilité du triangle qrs.

8. Pour la question qui nous occupe, ce n'est pas le nombre h lui-

même, caractéristique du mode B de dégénérescence, qui doit intervenir, mais deux nombres entiers, dont h est le quotient.

Ayant un système algébrique (a) de coniques a, on peut d'une infinité de manières trouver une courbe (k), telle que ses points k correspondent *uniformément* aux coniques a du système. M. Clebsch a, par exemple, démontré* que la courbe, lieu des pôles d'une droite fixe par rapport aux coniques a satisfait à cette condition. Ayant une pareille courbe, on peut lui substituer une quelconque de celles qui lui correspondent *point par point*, et varier ainsi à l'infini le mode de représentation du système (a) au moyen d'une courbe (k).

Soit K un point d'une telle courbe (k), et, s'il est singulier, considérons une des branches superlinéaires ou linéaires, disons abréviativement un des *cycles* (k_1), entre lesquels se répartissent les branches de la courbe en ce point. Si le point k se déplace sur ce cycle (k_1) aux environs de K, la conique a du système (a) varie aux environs de la conique limite A_1 qui correspond à K. Aux divers cycles dont l'origine est en K peuvent correspondre des limites diverses A de la conique a, comme aussi ces limites peuvent coïncider. De même aussi une pareille conique A peut correspondre à divers points K de la ligne (k). Mais ces diverses circonstances n'importent pas pour la question actuelle : le point k variant sur deux cycles différents (k_1), (k_2) de la courbe (k), la conique a aura deux limites A_1, A_2 qui, même si elles coïncident, devront être envisagées comme distinctes.

Supposons maintenant que A_1 soit une conique dégénérée suivant le mode B, et qu'ainsi deux des coefficients de (2), g_1, g_2, soient nuls. Soit λ l'ordre de multiplicité du cycle (k_1). Plaçons k sur ce cycle à une distance de K infiniment petite d'ordre λ. Pour ce point k, g_1 et g_2 sont infiniment petits. Soient m et $m+n$ leurs ordres respectifs. Ces deux nombres sont entiers, d'après la théorie des fonctions algébriques. Le quotient $n : m$ n'est autre que le nombre h ci-dessus défini. Mais les nombres m, n eux-mêmes sont caractéristiques de la conique dégénérée A_1. En effet, ils restent inaltérés si l'on change arbitrairement la courbe (k), comme cela résulte de la proposition suivante,† appartenant à la théorie générale des fonctions algébriques :

Soient (k), (k') *deux courbes algébriques se correspondant* point par point ; *à un cycle* (k_1) *de l'une correspond un seul cycle* (k_1') *de l'autre. Soient* λ, λ' *les ordres de multiplicité de deux cycles correspondants : à un point placé sur* (k_1) *à distance infiniment petite d'ordre* λ *de l'origine de ce cycle correspond un point placé sur* (k_1') *à distance infiniment petite d'ordre* λ' *de l'origine de ce dernier cycle.*

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* Mathematische Annalen, T. vi. p. 4.

† Bulletin de la Société Math. de France, T. iv. p. 32.

J'appellerai le nombre *m l'ordre* et le nombre *n la classe* de la conique dégénérée. On peut résumer comme il suit la définition de ces deux nombres :

Soit (k) *une courbe dont les points* k *correspondent* uniformément *aux coniques* a *d'un système* (a). *Soit* K *un point de cette courbe, et soit aussi* (k_1) *un cycle de branches ayant son origine en* K, *et d'ordre de multiplicité* λ. *Prenons, sur ce cycle, à distance infiniment petite l'ordre* λ *un point* k, *et soit* a *la conique qui correspond à ce point. Soit* $\frac{1}{2}m$ *l'ordre infinitésimal du segment intercepté par* a *sur une droite arbitraire, et* $\frac{1}{2}n$ *l'ordre infinitésimal de l'angle des tangentes menées à* a *par un point arbitraire. Les nombres* m, n *ne changent pas si l'on change la courbe* (k). *Ils seront appelés, le premier* l'ordre, *le second* la classe *de la conique dégénérée* a.

D'après cette définition, il n'est plus besoin de considérer trois modes de dégénérescence des coniques, mais un seul ; car *le mode* A *de dégénérescence correspond au cas où l'ordre est nul, et le mode* A′ *au cas où la classe est nulle.* Enfin, suivant le raisonnement du No. 6, on aperçoit que *dans deux systèmes corrélatifs, l'ordre d'une conique dégénérée de l'un quelconque d'entre eux est égal à la classe de la conique dégénérée correspondante dans l'autre.*

9. Pour bien préciser ces notions, prenons immédiatement un exemple que suggère naturellement le mode actuel d'exposition. Envisageons un système composé de coniques (2) conjuguées par rapport à un triangle fixe qrs, et, en outre, *unicursal ;* c'est-à-dire que les rapports des variables g sont des fonctions *rationnelles* d'une variable indépendante ω. Il est manifeste qu'on peut choisir cette variable ω de telle sorte, 1°, qu'en égalant chacun des g à un polynôme entier en ω, les trois polynômes soient d'un même degré ; 2°, qu' à chaque système de valeurs des g ne corresponde qu'une seule valeur de ω. Enfin il ne doit exister aucune racine commune à la fois aux trois polynômes, puisqu' il ne s'agit que des rapports des quantités g.

Ceci entendu, on aperçoit immédiatement qu'à une racine n'appartenant qu'à un seul polynôme correspond une conique dont la classe est précisément l'ordre de multiplicité de cette racine, et dont l'ordre est nul, ou, en d'autres termes, une conique dégénérée suivant le mode A. A une racine commune à deux polynômes et y ayant le même ordre de multiplicité, correspond une conique dégénérée suivant le mode A′, c'est-à-dire dont la classe est nulle et dont l'ordre est égal à l'ordre de multiplicité de la racine. Enfin, d'une manière générale, à une racine multiple d'ordre m dans un des polynômes, et d'ordre $m+n$ dans un autre, correspond une conique dégénérée d'ordre m et de classe n.

Dans cet exemple où l'on peut créer des coniques dégénérées en aussi grand nombre qu'on voudra, il est à remarquer que toutes celles

du mode B se superposeront à un nombre limité d'entre elles, sans que cependant il y ait aucune difficulté à les distinguer les unes des autres.

III. *Nombres caractéristiques relatifs à une condition pour une conique.*

10. Ainsi que je l'ai montré plus haut (No. 2), une condition quelconque Φ pour une conique peut être envisagée comme une relation numérique entre cette conique et quatre points t, u, v, w, cette relation étant, en outre, projective. Supposons-la exprimée au moyen des coefficients de l'équation ponctuelle de la conique. Alors Φ est représentée par l'évanouissement d'un covariant de a_x^2, contenant les coordonnées de quatre points indépendants. Quelques-uns de ces points peuvent y manquer. S'ils manquent tous, on a un invariant de a_x^2, c'est-à-dire son discriminant. Je le mets de côté, et il demeure entendu qu'il ne s'agit ici que de covariants. En outre, l'équation $\Phi=0$, entière par rapport aux coordonnées de t, ... w et aux coefficients de a_x^2, doit être indécomposable en facteurs de même forme. En d'autres termes, il s'agit de conditions indécomposables.

Dans l'équation $\Phi = 0$ supposons que le triangle de référence soit conjugué par rapport à la conique a, c'est-à-dire supprimons tous les termes contenant les coefficients des rectangles, a_{12}, ... Comme ci-dessus, appelons g les coefficients des trois carrés ; groupons les termes par rapport à ces quantités ; designant par Φ_1 ce que devient Φ, écrivons alors :

(3)..... $\Phi_1 = \Sigma H g_1^{\pi} g_2^{\chi} g_3^{\sigma}$.

Les coefficients tels que H sont des fonctions des coordonnées de t, u, v, w. Comme Φ est un covariant, il est visible que Φ_1 est symétrique par rapport aux g, *quant aux exposants*, c'est-à-dire qu'au terme mis en évidence dans (3) correspondent, comme existant effectivement, ceux qui s'en déduisent par la permutation des exposants, le coefficient H étant d'ailleurs changé. Considérons l'ensemble de ces divers termes, et caractérisons-le par les exposants, en l'appelant le *groupe* (π, χ, σ), et convenant de supposer $\pi \geq \chi \geq \sigma$. Entre ces trois nombres existe d'ailleurs la relation

(4)..... $\pi+\chi+\sigma = c =$ constante.

Ce sont ces groupes d'exposants qui vont jouer ici le rôle prépondérant, et je dois entrer dans quelques détails à leur sujet.

11. Parmi les groupes (π, χ, σ), il en est un au moins pour lequel σ est nul. Sans quoi, Φ_1 contiendrait le facteur $g_1 g_2 g_3$, et, par suite, Φ contiendrait en facteur le discriminant de a_x^2, ce qui est contre l'hypothèse.

En même temps que Φ, considérons une condition corrélative Φ'. On pourra exprimer cette condition en remplaçant dans Φ les coefficients de l'équation ponctuelle de la conique par ceux de l'équation tangentielle, en même temps que les coordonnées de t, u, v, w par les coordonnées de quatre droites. On aura donc Φ_1' en remplaçant, dans Φ_1, les g par leurs inverses, et chassant les dénominateurs. Pour obtenir le résultat définitif, imaginons d'abord que nous multipliions par $(g_1 g_2 g_3)^c$. Alors au terme $g_1^\pi g_2^\chi g_3^\sigma$ correspond le terme $g_1^{c-\pi} g_2^{c-\chi} g_3^{c-\sigma}$. Si Π est le maximum de π, l'équation obtenue contient alors le facteur $(g_1 g_2 g_3)^{c-\Pi}$. Ce facteur doit être supprimé. Si l'on pose alors :

$$\beta = \text{minimum de } (\chi+\sigma), \quad \Pi = c-\beta,$$

on trouve qu'au groupe (π, χ, σ) correspond le groupe (π', χ', σ') ainsi défini :

$$(5) \ldots\ldots \begin{cases} \pi' = \pi+\chi-\beta, \\ \chi' = \pi+\sigma-\beta, \\ \sigma' = \chi+\sigma-\beta, \end{cases} \qquad \begin{array}{c} c' = \pi'+\chi'+\sigma' = 2c-3\beta, \\ \pi' \geq \chi' \geq \sigma'. \end{array}$$

On passera de même de (π', χ', σ') à (π, χ, σ) par des équations analogues :

$$(6) \ldots\ldots \begin{cases} \pi = \pi'+\chi'-\alpha, \\ \chi = \pi'+\sigma'-\alpha, \\ \sigma = \chi'+\sigma'-\alpha, \end{cases} \qquad \begin{array}{l} c' = 2\alpha+\beta, \\ c = 2\beta+\alpha. \end{array}$$

Ces équations conduisent à définir autrement les groupes d'exposants, en introduisant les nombres qui s'échangent entre eux par dualité. On considérera pour Φ les deux nombres α, β et pour chaque groupe de termes les deux nombres σ, σ'. On dira alors, au lieu du groupe (π, χ, σ), le groupe (σ, σ'). On aura d'ailleurs d'après (5) et (6)

$$(7) \ldots\ldots \chi = \sigma'-\sigma+\beta, \qquad \pi = \alpha+\beta-\sigma',$$

et on remarquera que la condition $\sigma \geq \chi \geq \sigma$ donne :

$$(8) \ldots \beta \geq \text{maximum de } (2\sigma-\sigma'), \quad \alpha \geq \text{maximum de } (2\sigma'-\sigma).$$

12. Parmi les groupes (σ, σ'), quelques-uns sont à distinguer comme devant seuls jouer un rôle. Ce sont ceux que l'on peut appeler *minima*. Je dirai qu'un groupe (σ_1, σ_1') est *non-minimum* s'il en existe un autre (σ, σ') tel que l'on ait à la fois

$$\sigma < \sigma_1, \qquad \sigma' < \sigma_1',$$

une des deux inégalités pouvant se réduire à l'égalité. Tous les groupes non-minima éliminés, il en reste quelques-uns. Ce sont les derniers que j'appelle *minima*. On peut les obtenir ainsi : rangeons les divers groupes dans l'ordre croissant relativement à σ. Parmi ceux où σ est le même, prenons seulement celui où σ' est le plus petit, et omettons les autres. Puis, dans ceux qui restent, ne retenons que ceux qui se trou-

vent, à partir du premier, rangés par rapport à σ' dans l'ordre décroissant. Les groupes qui restent sont minima. Comme on l'a observé (No. 11), une des valeurs de σ est zéro. Il en est de même de σ'. Les groupes minima ainsi rangés forment donc une suite telle que

$$(9)\ldots\ldots\ldots\ldots (0\sigma'_0)\,(\sigma_1\sigma'_1)\,(\sigma_2\sigma'_2)\ldots(\sigma\sigma')\ldots(\sigma_k 0),$$
$$\sigma_i < \sigma_{i+1}, \quad \sigma'_i > \sigma'_{i+1},$$

et, en outre, les inégalités (8) ont lieu.

13. Je vais maintenant montrer que *si l'on donne à volonté la suite* (9) *et les deux nombres* α, β *satisfaisant aux inégalités* (8), *il existe des covariants* Φ *ayant les nombres caractéristiques* α, β *et les groupes minima donnés.*

Je prouve cette proposition en formant un tel covariant Φ comme il suit. Soit $P(\rho)$ une fonction homogène, de degré ρ, des coefficients de a_x^2, et à coefficients arbitraires. Soit de même $\Pi(\rho)$ une fonction homogène, de degré ρ, des coefficients de l'équation tangentielle de la conique, équation dont le premier membre est représenté par le symbole $(ab\xi)^2$. Soit enfin D le discriminant de a_x^2. Pour chacun des groupes donnés (9), par exemple pour le groupe (σ, σ'), considérons la fonction

$$(10)\ldots\ldots H(\sigma,\sigma') = D^\sigma \cdot \Pi(\beta - 2\sigma + \sigma') \cdot P(\alpha - 2\sigma' + \sigma),$$

dans laquelle, d'après (8), les degrés sont positifs, comme il convient. Pour chacun des groupes, nous varions, bien entendu, à volonté les coefficients des fonctions P et Π. Je fais maintenant

$$\Phi = \Sigma H(\sigma,\sigma'),$$

et je dis que la condition $\Phi = 0$ satisfait aux conditions demandées.

1°. Φ *est homogène par rapport aux coefficients de* a_x^2. Car D est du 3me degré, les coefficients de l'équation tangentielle sont du 2d degré; donc le degré de $H(\sigma,\sigma')$ est:

$$(11)\ldots\ldots 3\sigma + 2(\beta - 2\sigma + \sigma') + \alpha - 2\sigma' + \sigma = 2\beta + \alpha.$$

Donc $\Phi = 0$ exprime une condition pour la conique a_x^2.

2°. Un changement de coordonnées tel que (1) n'altère pas la forme des fonctions envisagées, mais rend seulement leurs coefficients des fonctions de t, u, v, w. *La forme de l'équation* $\Phi = 0$ *est donc inaltérée par une substitution homographique.*

3°. *Les groupes minima de* Φ *coïncident avec la suite* (9), *et les nombres caractéristiques sont* α, β. C'est là le point le plus important à prouver. L'égalité (11) prouve déjà que le degré de Φ est $(2\beta + \alpha)$. Si maintenant on remplace dans Φ les coefficients a de l'équation ponctuelle par ceux α de l'équation tangentielle, il faut comme on sait remplacer

$$\begin{array}{lccc} & a_{ij}, & \alpha_{ij}, & D \\ \text{par} & a_{ij}, & D\alpha_{ij}, & D^2. \end{array}$$

Les degrés de P et Π dans (10) s'échangent entre eux, et D est affecté de l'exposant $(\beta+\sigma')$, dont le minimum est β. Le facteur D^β supprimé, on a le degré de Φ' en transposant les lettres dans celui de Φ. Donc le degré c' de Φ' est $(2\alpha+\beta)$. Donc α, β sont bien les nombres caractéristiques de Φ que définissent les équations (5) et (6).

En ce qui concerne les groupes minima de Φ, annulons, conformément au No. 10, les coefficients des rectangles de a_x^2, et remplaçons les coefficients des carrés par les lettres g. Au lieu de (10) nous pouvons alors écrire :

$$(11)\ldots\ H_1(\sigma_1\sigma') = (g_1g_2g_3)^\sigma\,(g_1g_2,\ g_2g_3,\ g_3g_1)^{(\beta-2\sigma+\sigma')}\times(g_1,\ g_2,\ g_3)^{(\alpha-2\sigma'+\sigma)}.$$

Dans ce produit il existe évidemment un groupe de termes dans lesquels les exposants sont, en ordre croissant, σ, $\beta-\sigma+\sigma'$, $\alpha+\beta-\sigma'$, c'est-à-dire, d'après (7), un groupe (σ, σ'). Il reste à montrer que, relativement à tous les autres groupes déduits du même produit, le groupe (σ, σ') est minimum. Or deux lettres au moins, dans un terme quelconque, ont simultanément des exposants non inférieurs à la somme $(\beta-\sigma+\sigma')$ des deux premiers exposants dans (11). Soit donc (σ_1, σ_1') un autre groupe déduit de (11). L'exposant moyen χ, dans ce groupe, est, suivant (7), $\beta-\sigma_1+\sigma_1'$. Il n'est pas moindre que $\beta-\sigma+\sigma'$. Donc :

$$\sigma_1'-\sigma_1 \geq \sigma'-\sigma.$$

Mais aucune lettre ne peut avoir un exposant moindre que σ. Donc σ_1 est au moins égal à σ. Donc aussi, d'après la dernière inégalité, σ_1' n'est pas moindre que σ'. Donc le groupe (σ_1, σ_1') différent de (σ, σ') n'est pas minimum. Donc (σ, σ') est le seul groupe minimum provenant de $H(\sigma, \sigma')$; ce qui démontre la proposition annoncée.

14. On peut former d'une manière plus géométrique un covariant Φ ayant des groupes minima choisis à volonté. A cet égard, je me contenterai ici d'un énoncé facile à démontrer au moyen des résultats du No. 13.

Soit, sur une droite L, trois points donnés y, z, t ; soient x, x' les points où L est rencontrée par une conique a. Considérons les deux rapports anharmoniques (y, z, t, x) et (y, z, t, x'), et soit r leur différence.

Soit aussi, par un point l, trois droites données Y, Z, T ; soient X, X' les tangentes menées par l à a. Considérons de même la différence R des deux rapports anharmoniques (Y, Z, T, X), (Y, Z, T, X').

Soit maintenant, pour une conique a, la condition

$$\Phi = f(r^2, R^2) = \Sigma A r^{2\tau'} R^{2\tau} = 0,$$

f étant un polynôme entier. Les groupes d'exposants minima (σ, σ') de la condition Φ coïncident avec les groupes minima formés avec les nombres τ et τ'. Les nombres α, β sont égaux respectivement au double du maximum de τ' et de τ.

On peut, dans l'énoncé de cette dernière condition, substituer à r la longueur de la corde interceptée par a sur L, et à R le sinus ou la tangente trigonométrique de l'angle des tangentes issues de l. Les conclusions ne seront pas altérées. Mais, de la sorte, la condition n'est plus énoncée sous forme projective.

15. Revenons maintenant à la considération d'un covariant Φ quelconque, et de la fonction Φ_1 qui s'en déduit (No. 10). Les coefficients H de cette fonction dépendent des arbitraires t, u, v, w. Si, comme on en est convenu (No. 3), on laisse ces arbitraires indéterminées, les quantités H jouissent de la propriété suivante, qu'il est essentiel de mentionner pour la rigueur des démonstrations subséquentes :

Entre les quantités H il n'existe aucune relation linéaire et homogène identique

$$(12) \quad \ldots\ldots\ldots\ldots\ldots pH + p'H' + p''H'' + \ldots = 0,$$

dans laquelle $p, p', p''\ldots$ soient des constantes indépendantes de t, u, v, w.

Pour le prouver, rappelons-nous comment Φ a été supposé déduit d'une équation quelconque à laquelle doivent satisfaire les coefficients de l'équation d'une conique. C'est en faisant dans cette équation le changement de coordonnées (1). Or il est visible que ceci revient à remplacer dans les coefficients a_{ij}, symboliquement écrits $a_i a_j$, les symboles

	a_1,	a_2,	a_3
par	$(tvw)\, a_u$,	$(twu)\, a_v$,	$(tuv)\, a_w$.

Puis, pour obtenir Φ_1, on supprime les termes qui contiennent les coefficients des rectangles, et on remplace les coefficients des carrés par les lettres g. D'après cette observation, on se convaincra aisément que la quantité H, coefficient du terme mis en évidence dans (3), est par rapport aux lettres u, v, w affectées de l'indice 1 (en appelant u_1, u_2, u_3 les coordonnées de $u, \ldots$), homogène et du degré $2c+2\pi$; par rapport à ces mêmes lettres, affectées de l'indice 2, du degré $2c+2\chi$; et par rapport à ces lettres affectées de l'indice 3, du degré $2c+2\sigma$. Donc les coefficients H de deux termes qui diffèrent entre eux par les exposants des g, diffèrent eux-mêmes par leur composition en égard aux indices des lettres u, v, w. Donc l'identité (12) est impossible.

IV. *Du nombre des coniques d'un système, qui satisfont à une condition.*

16. Comme je l'ai annoncé au No. 4, je vais traiter le problème pour un cas particulier. La condition Φ sera quelconque, mais le système sera un de ceux dont la définition a été donneé au No. 9 ; il sera composé de coniques conjuguées par rapport à un triangle fixe, et unicur-

sal. Les trois quantités g sont alors des polynômes entiers d'une variable ω, et d'un même degré. Remplaçant les g par ces polynômes dans Φ_1, j'ai un nouveau polynôme entier en ω, dont les racines répondent aux coniques du système qui satisfont à Φ. La question proposée (No. 3) est donc ici ramenée à la discussion des racines de ce polynôme.

J'examinerai d'abord quelles sont les racines indépendantes des arbitraires t, u, v, w de la condition Φ, et quel est leur nombre. Je démontrerai ensuite que les autres racines sont simples ; elles répondent, comme on sait, aux véritables solutions cherchées. Le nombre de ces racines sera donc celui des solutions. J'obtiendrai d'ailleurs le nombre de ces racines comme la différence entre le degré du polynôme et le nombre des racines de la 1ère catégorie.

17. Soit Ω une racine du polynôme, que je continuerai à désigner par Φ_1 ; et supposons que cette racine ne varie pas avec t, u, v, w. En désignant par p la valeur de $g_1^\pi g_2^\chi g_3^\sigma$ (No. 10) pour $\omega = \Omega$, et de même par p', p'' ... les valeurs des autres termes, on a, d'après l'hypothèse :

$$pH+p'H'+p''H''+\ldots = 0,$$

qui doit être une identité. D'après le No. 15, cette identité est impossible. Donc p, p', p'' ... sont nuls séparément. Mais, parmi ces quantités, il en est qui ne contiennent pas le facteur $g_1g_2g_3$ (No. 11). Donc deux au moins des quantités g sont nulles. D'ailleurs les trois g ne sont pas nuls à la fois (No. 9). Donc la racine Ω correspond à une conique dégénérée dans le mode A' ou le mode B, ce qui est conforme aux observations du No. 4. Il s'agit maintenant de calculer l'ordre de multiplicité de cette racine. Soient m, $m+n$ les ordres de multiplicité de la racine Ω dans deux des polynômes g. Parmi les termes du groupe (π, χ, σ) (No. 10), celui qui contient le facteur $(\omega-\Omega)$ avec l'exposant le moindre, le contient au degré $\sigma(m+n)+\chi m$. Si maintenant, comme au No. 12, on introduit au lieu de π, χ, σ les nombres α, β, σ, σ', on a d'après (7) :

$$\sigma(m+n)+\chi m = \sigma(m+n)+(\sigma'-\sigma+\beta)m = \sigma n+\sigma' m+\beta m.$$

Si donc, considérant les divers groupes (σ, σ') de Φ_1, on pose

(13)..... $\gamma =$ minimum de $(\sigma n+\sigma' m)$,

on en conclut que Φ_1 contient le facteur $(\omega-\Omega)$ au moins au degré $\beta m+\gamma$. Mais je dis aussi que Φ_1 ne contient pas ce facteur à un degré plus élevé. En effet, ce facteur étant supprimé $(\beta m+\gamma)$ fois, il reste dans Φ_1 des termes qui séparément ne contiennent plus le facteur $(\omega-\Omega)$. L'ensemble de ces termes ne peut pas le contenir de nouveau ; car en faisant $\omega = \Omega$, on aurait encore une identité telle que (12), et

qui est impossible, quand t, u, v, w restent indéterminés. Ainsi l'ordre de multiplicité de la racine Ω dans Φ_1 est précisément $\beta m+\gamma$.

Que l'on prenne successivement chacune des racines Ω qui appartiennent à deux des polynômes g, et que pour chacune d'elles on forme le nombre analogue à γ, que l'on fasse ensuite la somme Γ de ces nombres γ; que l'on désigne ensuite par M la somme des nombres m, c'est-à-dire la somme des ordres de multiplicité de ces racines Ω pris pour chacune d'elles dans celui des deux polynômes g où cette racine entre effectivement, mais le moins de fois; la somme des ordres de multiplicité de ces racines Ω dans Φ_1 est égale à $\beta M+\Gamma$.

On remarquera: 1°, que les nombres m, n sont l'ordre et la classe de la conique dégénérée qui répond à $\omega = \Omega$; 2°, que les groupes minima (σ, σ') de Φ_1 concourent seuls à former les nombres tels que γ, ce qui justifie l'introduction des groupes minima dont j'ai parlé plus haut; 3°, parmi les valeurs de σ, σ' il y a toujours les valeurs zéro. Donc, si l'un des nombres m, n est nul, γ l'est aussi, c'est-à-dire que les coniques dégénérées suivant le mode B concourent seules à la formation du nombre Γ; 4°, enfin, si Φ_1 ne contient qu'un seul groupe minimum, qui est alors $(0, 0)$, γ est toujours nul, et par suite aussi Γ.

18. Démontrons maintenant que les autres racines du polynôme entier Φ_1 sont simples. S'il en était autrement, ce polynôme pourrait se décomposer en $\Psi^2\Theta$, Ψ et Θ ayant pour coefficients des fonctions entières des coefficients de Φ_1, par suite des fonctions entières des coordonnées de t, u, v, w. Ainsi Φ_1 se décomposerait de la sorte toutes les fois qu'on y mettrait pour la conique qui y figure une conique du système envisagé; par suite, en raisonnant comme au No. 4, on en conclurait que cette décomposition a lieu pour une conique quelconque. Donc la condition Φ serait décomposable, ce qui est contre l'hypothèse.

19. Les trois polynômes g sont, par hypothèse, d'un même degré a. En conséquence, le degré le plus élevé des termes de Φ_1 est $a(\pi+\chi+\sigma)$ ou (Nos. 10 et 11) $a(2\beta+\alpha)$. C'est précisément le degré de Φ_1; car un abaissement de ce degré exigerait une identité telle que (12) et qui est impossible. Donc le nombre des racines variables avec t, u, v, w est:

$$N = a(2\beta+\alpha)-\beta M-\Gamma = \alpha a+\beta(2a-M)-\Gamma.$$

Pour obtenir la signification géométrique des nombres a et $(2a-M)$ qui dépendent du système seul, il suffit de choisir deux exemples:

1°. Soit Φ la condition de passer par un point u. On a alors:

$$\Phi_1 = g_1u_1^2 + g_2u_2^2 + g_3u_3^2.$$

Ici (Nos. 10 et 11) $\alpha = 1$, $\beta = 0$; et il y a un seul groupe d'exposants

minima (0, 0). Donc $\Gamma = 0$. Si donc μ est le nombre des coniques du système qui passent par le point *arbitraire* u, on a $\mu = a$.

2°. Envisageons maintenant la condition corrélative, toucher une droite. D'après le No. 11, il suffit d'échanger α, β et σ, σ'. Donc $\Gamma = 0$, $\alpha = 0$, $\beta = 1$. Donc, si ν est le nombre des coniques qui touchent une droite arbitraire, on a $\nu = 2a - M$. La formule devient ainsi

$$(14) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad N = a\mu + \beta\nu - \Gamma,$$

et satisfait explicitement au principe de dualité. Dans cette formule, les éléments relatifs à la condition Φ ont été ici définis pour une condition quelconque. La démonstration précédente a traît à un système particulier. Mais j'ai donné (Nos. 7 et 8) les définitions des nombres m, n relatifs à une conique dégénérée d'un système quelconque ; d'autre part, les *caractéristiques* μ, ν ont une définition qui s'applique aussi à un système quelconque. Il n'y a donc aucune difficulté à appliquer à un tel système la formule (14). J'énonce ici, sans le démontrer, que cette application est légitime, et je renvoie, comme je l'ai dit précédemment, pour cette extension à un mémoire plus étendu qui sera publié ultérieurement.

20. Deux cas particuliers de la formule (14) sont à signaler, en se conformant aux remarques du No. 17.

Théorème I.—*Si un système de coniques, dont les caractéristiques sont* μ, ν, *ne contient aucune conique dégénérée suivant le mode* B, *le nombre des coniques de ce système qui satisfont, d'autre part, à une condition projective quelconque, dont les arbitraires restent indéterminées, est* $a\mu + \beta\nu$, α *et* β *étant deux nombres entiers qui ne dépendent que de cette condition.*

Remarque.—Les nombres α, β ont été définis par une représentation algébrique de la condition (No. 11) ; mais le théorème I en donne maintenant une définition géométrique, que l'on obtiendra en appliquant ce théorème à la condition envisagée et à deux systèmes différents, qui n'aient point de coniques dégénérées du mode B.

Le second cas particulier à signaler est celui où la fonction Φ_1 ne contient qu'un groupe minimum (0, 0). Dans ce cas, la formule (14) se réduit encore à $a\mu + \beta\nu$, *quel que soit le système.* Pour substituer à ce critérium algébrique un critérium géométrique, il suffit de considérer un système particulier contenant une conique dégénérée du mode B. En se plaçant à ce point de vue, on verra aisément que l'on peut dire :

Théorème II.—*Soit* S *le faisceau des coniques qui ont, en un point donné, des contacts du* 3^{me} *ordre avec une courbe donnée. Si le nombre des coniques de ce faisceau, qui satisfont à une condition* Φ *est* $\alpha + \beta$, *le nombre des coniques d'un système quelconque* (μ, ν) *qui satisfont à cette même condition* Φ *est toujours* $a\mu + \beta\nu$.

Ajoutons maintenant que : *Si un système S envisagé ne satisfait pas à la restriction énoncée au Th. I, et si une condition* Φ *ne satisfait pas à la restriction énoncée au Th. II, le nombre des coniques de S qui satisfont à* Φ *est inférieur à* $a\mu+\beta\nu$, *et en diffère d'un terme complémentaire* Γ.

21. Il reste maintenant à donner à ce terme complémentaire Γ une forme géométrique. Imaginons une fonction entière Ψ de deux variables x, y, ainsi composée

$$\Psi = \Sigma A x^{\sigma} y^{\sigma'},$$

les coefficients A y étant arbitrairement choisis, et les exposants σ, σ' étant les exposants minima de Φ_1. L'équation $\Psi = 0$ représente, en coordonnées rectilignes, une courbe qui passe à l'origine des coordonnées, sauf au cas où existe le groupe $(0, 0)$. Dans tout autre cas, la nature de ses branches à l'origine des coordonnées définit complètement les groupes minima (σ, σ'), et en donne une *image*. Appelons cette courbe Ψ *attachée* à la condition Φ.

Envisageons maintenant le cycle de branches de courbe représentés par les équations :

$$(15) \quad \ldots\ldots\ldots\ldots\ldots\ldots x = \omega^n f(\omega), \quad y = \omega^m \phi(\omega),$$

où f et ϕ sont des développements arbitraires suivant les puissances entières et ascendantes de ω, commençant par des termes de degré zéro. Le nombre des intersections des branches de ce cycle et de Ψ, réunies à l'origine des coordonnées est évidemment égal au minimum de $(\sigma n+\sigma' m)$. C'est justement l'expression (13) de l'élément γ. Composons donc une courbe Σ qui, pour chaque conique dégénérée, d'ordre m et de classe n, d'un système S, contienne un cycle de branches tel que (15). Alors le terme complémentaire Γ sera représenté par le nombre des points qu'ont en commun, à l'origine des coordonnées, les courbes Ψ et Σ. On peut varier à l'infini les courbes Ψ et Σ pour une même condition Φ et un même système S, puisque c'est la nature de leurs branches autour d'un point qui seule intervient. Je prendrai, par exemple, les définitions suivantes :

Pour chaque conique d'un système considérons les deux nombres R^2 *et* r^2 *définis dans l'énoncé du No.* 14, *et prenons-les, le premier pour l'abcisse, le second pour l'ordonnée d'un point. La conique variant dans le système, ce point décrit une courbe, que nous dirons* attachée au système. (On peut, suivant la remarque du No. 14, remplacer R et r par un sinus et une longueur.)

Soit une condition Φ *pour une conique. Supposons une conique conjuguée par rapport à un triangle abc ; soit p un des points où cette conique rencontre bc, et soit q un des points où elle rencontre ab. Pour cette*

conique, la condition peut s'exprimer par une relation $\psi(x, y) = 0$ *entre*

$$x = \left(\frac{qa}{qb}\right)^2, \quad y = \left(\frac{pb}{pc}\right)^2.$$

La courbe représentée par l'équation $\psi(x, y) = 0$ *sera dite* attachée à la condition Φ.

Ces définitions posées, on peut dire alors :

THÉORÈME III.—*Le terme complémentaire relatif à un système et à une condition est égal au nombre des points qu'ont en commun à l'origine des coordonnées la courbe attachée au système et la courbe attachée à la condition.*

V. *Exemples.*

22. Je vais donner ici quelques exemples du calcul de la formule (14) en envisageant des conditions Φ, données sous des formes géométriques.

Je suppose que la condition Φ consiste en une relation projective entre la conique a et une conique donnée b. Cette condition s'exprime, comme on sait, par une relation entre les coefficients du discriminant de $a+\lambda b$, et cette relation est homogène par rapport aux coefficients de a, ainsi que par rapport à ceux de b. Il en résulte aisément que cette relation peut aussi être envisagée comme une équation symétrique et homogène entre les trois racines $\lambda, \lambda', \lambda''$ de ce discriminant, et réciproquement. Je suppose la relation mise sous cette dernière forme, et, sans la transformer, je vais faire le calcul de la formule (14) relatif à cette condition et à un système arbitraire.

En premier lieu, le degré de l'équation, par rapport à $\lambda, \lambda', \lambda''$, est aussi celui de Φ_1. Soit c ce degré, on a déjà

$$2\beta + \alpha = c,$$

suivant les formules (6). Si a est une conique dégénérée, d'ordre m et de classe n, il est visible que deux des λ sont infiniment petits, l'un d'ordre m, l'autre d'ordre $m+n$, exactement comme les g (No. 8). Faisons donc λ infiniment petit d'ordre $m+n$, λ' infiniment petit d'ordre m. Si le premier membre de l'équation devient ainsi infiniment petit d'ordre p, on aura

$$p = \beta m + \gamma.$$

En choisissant de diverses manières les nombres m, n, on déterminera ainsi β et les groupes (σ, σ') de la condition.

Dans beaucoup de cas, la relation entre les λ n'est pas immédiatement donnée sous forme symétrique. Quand il en est ainsi, on la rend

symétrique en faisant le produit des facteurs dissymétriques différents, obtenus par la permutation des lettres. Il est clair qu'alors on peut faire porter le calcul sur un seul facteur. Ces généralités admises, je prends des exemples concrets.

23. Soit d'abord pour Φ la condition que a touche b. Cette condition s'exprime par $\phi = \lambda - \lambda' = 0.$

Il y a six facteurs analogues à multiplier entre eux. Donc :

$$2\beta + \alpha = 6.$$

On en pourrait déduire $\alpha = \beta = 2$; car il est manifeste que α et β sont des nombres égaux. Mais appliquons ici notre méthode. Le facteur ϕ n'est infiniment petit que si λ et λ' sont tous deux infiniment petits ; et si les ordres respectifs de λ et λ' sont $m+n$ et m, $\lambda - \lambda'$ est d'ordre m. Donc deux des six facteurs sont infiniment petits d'ordre m, les autres restent finis. Donc

$$\beta m + \gamma = 2m.$$

Donc γ est toujours nul, et β est égal à 2. Donc, suivant un théorème bien connu : *le nombre des coniques d'un système* (μ, ν) *quelconque qui touchent une conique donnée est égal à* $2(\mu + \nu)$. C'est un exemple du théorème II.

24. *Soit* r *le rapport anharmonique des quatre points d'intersection de* a *avec une conique fixe* b, *considérés sur* a ; *soit* s *le rapport anharmonique des quatre mêmes points, considérés sur* b, *dans le même ordre ; et envisageons successivement les deux conditions définies, pour la conique* a,

par les équations $\phi = s^p - kr^q = 0, \quad \psi = s^p r^q - k = 0,$

k *étant un nombre donné et* p, q *des entiers premiers entre eux.*

A l'égard de la condition ϕ, on trouvera sans peine, si $p > q$, que le facteur type est :

$$\phi = \lambda'^q (\lambda' - \lambda'')^{p-q} - k\lambda^q (\lambda - \lambda'')^{p-q}.$$

En raisonnant, comme au No. 23, on trouvera que

Si $p > q$, *le nombre des coniques d'un système* (μ, ν) *quelconque, qui satisfont à la condition* ϕ, *est* $(6p - 4q)\mu + 2q\nu$; et aussi,

Si $p < q$, *ce nombre se change en* $2p\mu + (6q - 4p)\nu$.

Ces deux nombres se réduisent de moitié si $k = -1$; *et, si* $k = 1$, *ils doivent être diminués de* $2(\mu + \nu)$ *et ensuite divisés par* 2. Ces deux circonstances proviennent de ce que, dans le premier cas, ϕ est

symétrique en λ, λ', et que, dans le second, il devient symétrique après suppression du facteur $(\lambda-\lambda')$.

Ainsi la condition ϕ donne encore exemple du théorème II.

Mais il n'en est plus de même de la condition ψ. Le facteur type

est alors $$\psi = (\lambda'-\lambda'')^{p+q}\lambda^q - k(\lambda-\lambda'')^{p+q}\lambda'^q,$$

et l'on en tire les conclusions suivantes :

Soient μ, ν les caractéristiques d'un système S, N' la somme des classes de celles des coniques dégénérées de S pour chacune desquelles le rapport de la classe à l'ordre est inférieur à $p:q$, et M'' la somme des ordres des autres coniques dégénérées. Le nombre des coniques de S qui satisfont à la condition Ψ est :

$$6(p\mu+q\nu)-2N'q-2M''p.$$

On voit qu'ici il y a un terme complémentaire $2(N'q+M''p)$ qui est nul dans le cas seulement où le système S ne contient aucune conique dégénérée suivant le mode B.

On remarquera encore que, *si $k=-1$, le résultat doit être divisé par* 2, *et, si $k=1$, il doit être diminué de* $2(\mu+\nu)$ *puis divisé par* 2.

25. Voici un dernier exemple où le calcul se fait d'une manière analogue. *On donne deux coniques fixes b, c, et l'on envisage, pour une conique a, la condition que le rapport anharmonique de ses points d'intersection avec b, pris sur a, soit égal au rapport anharmonique de ses points d'intersection avec c, pris sur c.*

Soient λ, λ', λ'' les racines du discriminant de $a+\lambda b$, et aussi l, l', l'' les racines du discriminant de $a+lc$. Le facteur est ici :

$$\phi = \lambda(\lambda'-\lambda'')(l-l'') - \lambda'(\lambda-\lambda'')(l'-l'').$$

Il faut permuter entre elles les lettres λ et aussi les lettres l, ce qui donne 36 combinaisons. Mais, ainsi que je vais le montrer, 12 seulement de ces combinaisons sont distinctes, en sorte qu'on aura :

$$2\beta+\alpha = 3\times 12 = 36,$$

d'où l'on pourrait déjà conclure $\alpha=\beta=12$.

Si l'on désigne par r, s les deux rapports anharmoniques, on obtiendra six facteurs distincts en donnant successivement à s six déterminations distinctes sans altérer r. Ces six facteurs dérivent donc de $(r-s)$ par le changement de s seulement. Puis en changeant successivement r on aura six groupes de six facteurs chacun. Mais trois de ces groupes contiennent l'expression $(r-s)$ et trois autres l'expression $(s-r)$, comme on le voit en faisant le même changement à la fois sur r et sur s dans $(r-s)$. Donc il y a en tout 12 facteurs, savoir les 6 fac-

teurs qu'on déduit de $(r-s)$ par le changement de s seul, puis 6 facteurs respectivement égaux et de signes contraires aux précédents. Il suffira donc d'opérer sur 6 facteurs déduits de $(r-s)$ par changement de s seulement, et de doubler les résultats. Ainsi je n'ai qu'à permuter les lettres l dans ϕ en supposant, par exemple :

λ, l infiniment petits d'ordre $m+n$,
λ', l' „ „ m,
λ'', l'', finis.

Il est évident que les facteurs seront d'ordre m, à moins que le second terme, qui contient déjà $\lambda'(\lambda-\lambda'')$, ne contienne $(l-l')$. Donc 4 facteurs sont d'ordre m, et il y a exception pour les deux suivants :

$$\phi_1 = \lambda(\lambda'-\lambda'')(l''-l') - \lambda'(\lambda-\lambda'')(l-l'),$$
$$\phi_2 = \lambda(\lambda'-\lambda'')(l''-l) - \lambda'(\lambda-\lambda'')(l'-l),$$

qui sont d'ordre $m+n$ ou d'ordre $2m$ suivant que n est inférieur ou supérieur à m. Delà enfin la conclusion suivante :

Soient, pour un système de coniques, μ, ν les caractéristiques, N' la somme des classes de celles des coniques dégénérées pour chacune desquelles la classe est inférieure à l'ordre, et M'' la somme des ordres des autres coniques dégénérées. Le nombre des coniques de ce système qui satisfont à la condition ci-dessus est $12(\mu+\nu)-4(M''+N')$.*

Si, au lieu de cette condition, on envisageait celle qui consisterait en ce que les deux mêmes rapports anharmoniques fussent égaux et *de signes contraires*, on verrait immédiatement que le résultat précédent devrait être simplement réduit de moitié.

26. Pour que l'on se rende compte des circonstances qui s'offrent dans les exemples précédents, j'ajoute ici quelques considérations géométriques au sujet du rapport anharmonique de quatre points, pris sur une conique qui dégénère.

Soient 1, 2, 3, 4 les points d'intersection de deux coniques a, b, et I, II, III, IV les points de contact de b avec les tangentes communes à a et b. On sait que le point de rencontre de deux cordes conjuguées quelconques du premier système est aussi un point de rencontre de deux cordes conjuguées du second. En conséquence, si à un point d'un des systèmes on associe un des points de l'autre, en le choisissant d'ailleurs à volonté, il en résulte une association parfaitement déterminée

* Dans ma communication du 13 Nov. 1876, adressée à l'Académie des Sciences, j'ai énoncé fautivement ce résultat, en le divisant à tort par 2.

entre les points des deux systèmes, association définie par cette condition que deux cordes conjuguées quelconques d'un des systèmes concourent avec leurs associées.

Ceci entendu, on a cette proposition : *le rapport anharmonique des quatre points* 1, 2, 3, 4, *considérés sur la conique a dans un ordre quelconque, est égal au rapport anharmonique des points* I, II, III, IV *sur la conique b, pris dans l'ordre correspondant par association.*

Supposons que a, faisant partie d'un système, dégénère suivant le 3me mode. Soient D et d la droite et le point, limites de a, le point d sur la droite D; p, q les points de rencontre de b et D; r, s les points de contact de b avec les tangentes issues de d. Les coniques a et b ont en commun deux points 1, 2 infiniment voisins de p, et deux points 3, 4, infiniment voisins de q. Leurs tangentes communes ont avec b deux points de contact infiniment voisins de r, et deux points de contact infiniment voisins de s. Si deux points associés portent le même numéro, on voit aisément que, suivant la disposition des points 1, 2, 3, 4, le point I étant censé voisin de r, ce sera en même temps le point III ou le point IV qui sera aussi voisin de r. Dans le premier cas, le rapport anharmonique

$$(\mathrm{I, II, III, IV}) = \frac{\mathrm{I\ II}}{\mathrm{I\ III}} \cdot \frac{\mathrm{III\ IV}}{\mathrm{II\ IV}}$$

croît au-delà de toute limite; dans le second, il a l'unité pour limite. En même temps, le rapport anharmonique associé (1, 2, 3, 4), considéré sur b, a zéro pour limite. Par suite, le tableau suivant montre que ces deux cas n'en font qu'un, et qu'aux deux valeurs zéro du rapport anharmonique de 1, 2, 3, 4 sur b correspondent les valeurs 1 et ∞ du rapport anharmonique de I, II, III, IV.

ω	$\frac{1}{\omega}$	$1-\omega$	$\frac{1}{1-\omega}$	$\frac{\omega-1}{\omega}$	$\frac{\omega}{\omega-1}$
0	∞	1	1	∞	0
1	1	0	∞	0	∞

Mais, d'après la proposition rappelée plus haut, les rapports anharmoniques de I, II, III, IV coïncident avec ceux de 1, 2, 3, 4 envisagés sur a. Donc on voit que le rapport anharmonique de ces derniers points, envisagé sur l'une des coniques, devenant zéro, il devient 1 ou ∞ quand on l'envisage sur l'autre. C'est là ce qui explique les circonstances rencontrées dans les exemples précédents.

VI. *Des équivalents.*

27. Jusqu'ici j'ai considéré exclusivement des conditions indécomposables. J'introduis maintenant des conditions composées. Soient

$\Phi' = 0$, $\Phi'' = 0$, ... diverses conditions. L'équation $0 = \Phi = \Phi'\Phi''\ldots$ définit une condition composée, que l'on pourra dire la *somme* des composantes pour la raison suivante. Soit N' le nombre des coniques d'un système (a) qui satisfont à Φ'; soit N'' le nombre des coniques de (a) qui satisfont à Φ''; ... soit enfin Φ le nombre des coniques de (a) qui satisfont à Φ; on a $N = N' + N'' + \ldots$

Soient Φ et Φ_1 deux conditions telles que le nombre des coniques d'un système (a) qui satisfont à l'une ou à l'autre soit le même, et que cette circonstance ait toujours lieu, quel que soit (a). Je dirai que Φ et Φ_1 sont deux conditions *équivalentes*.

28. Considérons, comme au No. 21, une courbe Ψ attachée à une condition Φ, supposons cette courbe décomposée, à l'origine des coordonnées, en ses branches linéaires ou superlinéaires, et soient

$$x = t^q, \quad y = t^p f(t)$$

les équations qui représentent une de ces branches. La lettre f désigne un développement suivant les puissances entières et ascendantes de t, commençant par un terme constant; et p et q sont deux nombres entiers positifs. De plus, les coefficients A de l'équation qui définit Ψ (No. 21) restant indéterminés, on sait par la théorie des courbes algébriques que p et q sont premiers entre eux. Soient (p, q), (p', q') ... les couples de nombres analogues, l'identité de deux couples étant d'ailleurs possible.

Pour l'objet actuel, on peut remplacer la courbe Ψ par la courbe composée

$$0 = \Psi_1 = (x^p - By^q)(x^{p'} - B'y^{q'}) \ldots\ldots$$

En effet, en vertu de l'indétermination des coefficients A, B, ... le nombre des intersections confondues à l'origine, avec une courbe quelconque, sera le même pour Ψ ou pour Ψ_1.

Or, d'après un résultat du No. 14, Ψ_1 est composée des courbes attachées aux diverses conditions

$$r^{2q} = CR^{2p}, \quad r^{2q'} = C'R^{2p'}, \quad \ldots\ldots$$

Désignons par (p, q) une telle condition *élémentaire*. Nous avons ce premier résultat, que la courbe attachée à Φ est la superposition ou la *somme* des courbes attachées à plusieurs conditions élémentaires.

D'après le No. 14 encore, les nombres analogues à α et β sont, pour la condition (p, q), respectivement $2q$ et $2p$. Donc, pour la condition Φ_1, somme de (p, q), (p', q'), ..., on a:

$$\alpha_1 = 2(q + q' + \ldots), \quad \beta_1 = 2(p + p' + \ldots).$$

Mais, suivant le No. 12, ces deux derniers nombres sont précisément égaux respectivement à $2\sigma'_0$ et $2\sigma_k$. Si l'on fait

$$a = \alpha - \alpha_1, \qquad b = \beta - \beta_1,$$

les inégalités (8) montrent alors que a et b sont des nombres positifs.

Designons par P et par D la condition, pour une conique, de passer par un point ou de toucher une droite; il résulte de ce qui précède l'équivalence $\Phi \equiv aP + bD + (p, q) + (p', q') + \ldots$.

En effet, la condition composée, représentée par le second membre, a la même courbe attachée et les mêmes nombres α, β que la condition Φ. Donc elle lui est équivalente. Donc

Théorème IV.—*Une condition quelconque est équivalente à la somme d'un nombre limité de conditions élémentaires.*

On peut dès lors remplacer les propositions précédentes par le Théorème IV et le suivant:

Théorème V.—*Soit, dans un système, de caractéristiques μ, ν, une conique dégénérée d'ordre m et de classe n; soit, pour cette figure, γ le plus petit des deux produits mq, np. Soit enfin Γ la somme des nombres analogues à γ pour toutes les coniques dégénérées du système. Le nombre des coniques de ce système qui satisfont à la condition (p, q) est* $2(q\mu + p\nu) - \Gamma$.

On pourra *conventionnellement* mettre au nombre des conditions (p, q) les deux conditions P, D, en supposant:

$$P \equiv (0, \tfrac{1}{2}), \qquad D \equiv (\tfrac{1}{2}, 0).$$

Pour tout autre cas, p, q seront des entiers premiers entre eux. La convention précédente a l'avantage de réduire à sa plus simple expression la formule qui donne le nombre des coniques satisfaisant à cinq conditions séparées, formule dont la recherche fait la suite naturelle du mémoire actuel.

29. Les exemples envisagés au paragraphe V revêtiront maintenant les formes suivantes:

Pour la condition ϕ du No. 24, on aura:

Si $p > q$, $\phi \equiv (6p - 4q)P + 2qD$.

Si $p < q$, $\phi \equiv 2pP + (6q - 4p)D$.

Pour la condition ψ de ce même No. 24, on a:

$$\psi \equiv 2pP + 2qD + 2(q, p);$$

et pour celle du No. 25,

$$\Phi \equiv 4P + 4D + 4(1, 1).$$

Ces exemples me paraissent suffire à éclaircir les théories exposées dans ce mémoire.

On Flexure of Spaces. By Mr. C. J. Monro, B.A.

[*Read June 13th*, 1878.]

[On the suggestion of Prof. Cayley, who acted as a referee to this paper, the following note by him is prefixed by way of illustration.

Consider an element of surface, surrounding a point P; the flexure of the element may be interfered with by the continuity round P, and it is on this account proper to regard the element as cut or slit along a radius drawn from P to the periphery of the element. This being understood, we have the well-known theorem that, considering in the neighbourhood of the origin elements of the surfaces

$$z = \tfrac{1}{2}(ax^2+2hxy+by^2), \quad \text{and} \quad z' = \tfrac{1}{2}(a'x'^2+2h'x'y'+b'y'^2),$$

these will be applicable the one on the other, provided only $ab-h^2 = a'b'-h'^2$. But in connexion with Mr. Monro's paper it is worth while to give the proof in detail.

It is to be shown that z, z' denoting the above-mentioned functions of (x, y) and (x', y') respectively, it is possible to find (for small values) x', y' functions of x, y such that identically

$$dx'^2+dy'^2+dz'^2 = dx^2+dy^2+dz^2.$$

The solution is taken to be $x' = x+\xi$, $y' = y+\eta$, where ξ, η denote cubic functions of x, y. We have then, attending only to the terms of an order not exceeding 3 in x, y,

$$dx^2+dy^2+2(dx\,d\xi+dy\,d\eta)+\{(a'x+h'y)\,dx+(h'x+b'y)\,dy\}^2$$
$$= dx^2+dy^2+\{(ax+hy)\,dx+(hx+by)\,dy\}^2;$$

so that the terms dx^2+dy^2 disappear, and then writing

$$d\xi = \frac{d\xi}{dx}dx+\frac{d\xi}{dy}dy, \qquad d\eta = \frac{d\eta}{dx}dx+\frac{d\eta}{dy}dy,$$

the equation will be satisfied identically as regards dx, dy if only

$$2\frac{d\xi}{dx} = (ax+hy)^2-(a'x+h'y)^2,$$

$$\frac{d\xi}{dy}+\frac{d\eta}{dx} = (ax+hy)(hx+by)-(a'x+h'y)(h'x+b'y),$$

$$2\frac{d\eta}{dy} = (hx+by)^2-(h'x+b'y)^2;$$

or, calling the terms on the right-hand side $2\mathfrak{A}$, $\mathfrak{B}$, $2\mathfrak{C}$ respectively, we have

$$\frac{d^2\mathfrak{A}}{dy^2}-\frac{d^2\mathfrak{B}}{dx\,dy}+\frac{d^2\mathfrak{C}}{dy^2} = 0,$$

that is,

$$(h^2-h'^2)+(h^2-h'^2)-\{(ab+h^2)-(a'b'+h'^2)\} = 0,$$

or, what is the same thing, $a'b'-h'^2 = ab-h^2$,

a relation which must exist between the constants (a, b, h) and (a', b', h').

It is easy to find the actual values of ξ, η; viz., these are

$$\xi = \tfrac{1}{6}(a^2 - a'^2)\,x^3 + \tfrac{1}{2}(ah - a'h')\,x^2y + \tfrac{1}{2}(h^2 - h'^2)\,x^2y + \tfrac{1}{6}(bh - b'h')\,y^3,$$
$$\eta = \tfrac{1}{6}(ah - a'h')\,x^3 + \tfrac{1}{2}(h^2 - h'^2)\,x^2y + \tfrac{1}{2}(bh - b'h')\,x^2y + \tfrac{1}{6}(b^2 - b'^2)\,y^3,$$

or, what is the same thing, we have $\xi = \dfrac{1}{24}\dfrac{d\Omega}{dx}$, $\eta = \dfrac{1}{24}\dfrac{d\Omega}{dy}$, where

$$\Omega = (a^2 - a'^2)\,x^4 + 4(ah - a'h')\,x^3y + 6(h^2 - h'^2)\,x^2y^2 + 4(bh - b'h')\,xy^3 + (b^2 - b'^2)\,y^4,$$
$$= (ax^2 + 2hxy + by^2)^2 - (a'x^2 + 2h'xy + b'y^2)^2 = 4(z^2 - z'^2),$$

in virtue of the relation $ab - h^2 = a'b' - h'^2$. The resulting values $x' = x + \xi$, $y' = y + \eta$ are obviously the first terms of two series which, if continued, would contain higher powers of (x, y).]

Space of n dimensions admits in general of pure flexure in space of $n+n'$ dimensions with not more than $\dfrac{n(n+1)}{2}\left(n' - \dfrac{n(n-1)}{6}\right)$ degrees of freedom.

By pure flexure I mean flexure without extension or compression. No portion of space admits of pure flexure except in space of a higher number of dimensions. This is why, in experience, a solid cannot be bent at all except so far as it is extensible or compressible; an inextensible and incompressible sheet, however flexible, cannot be bent in its own plane.

Flexure of a space of any number of dimensions is sufficiently defined by the condition that lines in it shall change curvature; pure flexure by the further condition that the shortest distance in the space between every two points shall remain constant.

I say "not more," assigning only a maximum freedom, because I shall examine only the conditions depending upon the nature of the space at any assumed point. It is to be remembered that the flexure of any portion of space depends partly upon the general configuration of its continuous extent; for instance, in space of three dimensions, a closed surface does not, *exceptis excipiendis*, admit of pure flexure at all.

It will be convenient to suppose the space of n dimensions defined by n' equations, which, allowing for terms of the third and higher orders, shall give each of n' coordinates as a quadric in the n others. These equations will be of the form

$$2z_h = \Sigma_i Z_h (ii)\,x_i^2 + 2\Sigma_{ij} Z_h (ij)\,x_i x_j + T \ldots\ldots\ldots\ldots (1),$$

where i and j range from 1 to n, and h from 1 to n', T being of the third order at lowest, and the Z's parameters of the space. Flexure will change the values of these parameters, but need not change the form of the equations; for we may suppose that there is no translation or rotation at the origin relatively to the axes of coordinates. Reduction to this form is always possible. After transformation to an origin in the space of n dimensions, we have $(n+n')^2$ quantities to dispose of in the formulas for transformation of direction: the first-degree x-terms are got rid of by nn' conditions, and in order to connect each z severally with the x's it is enough to eliminate the first powers of every $n'-1$ of them successively, for their higher powers will thus be thrown into T, where they will not disturb the form (1). We shall further be able to make the coordinates rectangular; for this will be done by $\frac{1}{2}(n+n')(n+n'+1)$ conditions, a number which, subtracted from $(n+n')^2-nn'$, leaves the positive value $\frac{1}{2}n(n-1)+\frac{1}{2}n'(n'-1)$.

The proposition announced depends partly on the number of our equations, but not otherwise upon their form. It is therefore independent of assumptions in general as to the nature of the space of $n+n'$ dimensions. Let us assume this to be "plane" (that is, such that its geodetic lines conform to Euclid's fifth and sixth postulates), and proceed metrically. If then we suppose the coordinates rectangular, and distinguish different points of the space in any of its states by δ, and different positions of the same point of the space in its different states by Δ, we must have $\Delta\delta s=0$, where $\delta s^2=\Sigma\delta x^2+\Sigma\delta z^2$, and therefore

$$\Sigma\delta x\,\delta\Delta x+\Sigma\delta z\,\delta\Delta z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Here $\delta x_i\,\delta\Delta x_i=\delta x_i\,\Sigma_j\frac{d\Delta x_i}{dx_j}\delta x_j$, $\delta z_h\,\delta\Delta z_h=\Sigma_i\frac{dz_h}{dx_i}\delta x_i\,\Sigma_j\frac{d\Delta z_h}{dx_j}\delta x_j$;

therefore, since the x's are independent, equation (2) requires a system of equations of the form

$$\frac{d\Delta x_i}{dx_j}+\frac{d\Delta x_j}{dx_i}+\Sigma_h\left(\frac{dz_h}{dx_i}\frac{d\Delta z_h}{dx_j}+\frac{dz_h}{dx_j}\frac{d\Delta z_h}{dx_i}\right)=0 \quad \ldots\ldots\ldots(3).$$

These are $\frac{1}{2}n(n+1)$ in number, n in which $j=i$, and $\frac{1}{2}n(n-1)$ in which i and j are different.

The Δ-variation of each z depends partly on variation of the x's, and partly upon variation of the Z's. Mark the latter part by Δ_a. Then

$$\Delta z_h=\Sigma_g\frac{dz_h}{dx_g}\Delta x_g+\Delta_a z_h$$

(where g ranges from 1 to n); and therefore

$$\frac{d\Delta z_h}{dx_i}=\Sigma_g\left(\frac{d^2z_h}{dx_i\,dx_g}\Delta x_g+\frac{dz_h}{dx_g}\frac{d\Delta x_g}{dx_i}\right)+\Delta_a\frac{dz_h}{dx_i}.$$

Here the second differential coefficient is, by equation (1), the para-

meter $Z_h(ig)$, and $\frac{dz_h}{dx_g}$ is of the first order; therefore such quantities as that under the sign Σ_g are of the order next above the first two terms of (3), and may be neglected upon substitution in (3), especially as they appear there multiplied by $\frac{dz_h}{dx_i}$ and $\frac{dz_h}{dx_j}$. We may write, then,

$$\frac{d\Delta x_i}{dx_j} + \frac{d\Delta x_j}{dx_i} + \Sigma_h \left(\frac{dz_h}{dx_i} \Delta_a \frac{dz_h}{dx_j} + \frac{dz_h}{dx_j} \Delta_a \frac{dz_h}{dx_i} \right) = 0,$$

or

$$\frac{d\Delta x_i}{dx_j} + \frac{d\Delta x_j}{dx_i} + \Sigma_h \Delta_a \frac{dz_h}{dx_i} \frac{dz_h}{dx_j} = 0,$$

which shows, be it said in passing, that the Δx's, hitherto quite undetermined, are of the third order, irrespectively of lower terms depending on the translation or rotation which we leave out of account. Two differentiations will eliminate the infinitesimals:

$$\frac{d^2\Delta x_i}{dx_j\, dx_k} + \frac{d^2\Delta x_j}{dx_i\, dx_k} + \Sigma_h \Delta_a \left(\frac{d^2 z_h}{dx_i\, dx_k} \cdot \frac{dz_h}{dx_j} + \frac{dz_h}{dx_i} \cdot \frac{d^2 z_h}{dx_j\, dx_k} \right),$$

$$\frac{d^3\Delta x_i}{dx_j\, dx_k\, dx_l} + \frac{d^3\Delta x_j}{dx_i\, dx_k\, dx_l} + \Sigma_h \Delta_a \left(\frac{d^2 z_h}{dx_i\, dx_k} \cdot \frac{d^2 z_h}{dx_j\, dx_l} + \frac{d^2 z_h}{dx_i\, dx_l} \cdot \frac{d^2 z_h}{dx_j\, dx_k} \right).$$

We may write this last equation thus:

$$jkl, i + ikl, j + ik, jl + il, jk = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

of which the first two terms are (but for a numerical factor) coefficients in the expressions for the Δx's, and as for the last two,

$$ik, jl = \Sigma_h \Delta Z_h(ik)\, Z_h(jl).$$

The desired conditions, that the new space shall be derived by pure flexure from the old, will be given by equations among the variations forming these sums. To obtain such equations we must eliminate the first two terms from the system represented by (4). This system consists of senary groups involving the same four elements. Thus, for i, j, k, l,

$$\begin{aligned}
&jkl, i + kli, j \qquad\qquad + ik, jl + il, jk = 0,\\
&jkl, i \quad + lij, k \qquad\qquad + il, jk + ij, kl = 0,\\
&jkl, i \qquad + ijk, l + ik, jl \qquad + ij, kl = 0,\\
&kli, j + lij, k \qquad + ik, jl \qquad + ij, kl = 0,\\
&kli, j \qquad + ijk, l \qquad + il, jk + ij, kl = 0,\\
&lij, k + ijk, l + ik, jl + il, jk \qquad = 0;
\end{aligned}$$

whence

$$\begin{aligned}
\tfrac{1}{2}(jkl, i + kli, j + lij, k + ijk, l) &= ik, jl + il, jk\\
&= il, jk + ij, kl\\
&= ik, jl \quad + ij, kl,
\end{aligned}$$

and, in short,

$$ik, jl = il, jk = ij, kl \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

There are $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$ such pairs of equations in which the four elements are distinct. Let $l=i$; then the two reduce to one,

$$ik, ij = ii, jk \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)};$$

but the same three elements yield three such equations, so that there are $\frac{1}{2}n(n-1)(n-2)$ such in all. Let $l=i$ and $k=j$; then

$$ii, jj = ij, ij \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(C)},$$

and of this type there are $\frac{1}{2}n(n-1)$.

Of the types (B) and (C) there are $\frac{1}{2}n(n-1)^2$, and of all three there are $\frac{1}{1\,2}n^2(n^2-1)$ equations.

Of the quantities entering into these equations, the Z's are known as parameters of the given space, and we have to determine their Δ's. There are $\frac{n(n+1)}{2}.n'$ of these, which therefore vary with $\frac{1}{2}n(n+1) \times [n' - \frac{1}{6}n(n-1)]$ degrees of freedom, as was to be proved.

It should be observed that the nn' quantities of the form $Z_h(ii)$ enter into only the $\frac{1}{2}n(n-1)^2$ equations of the types (B) and (C). But these equations are enough to determine that class of quantities, unless n' is greater than $\frac{1}{2}(n-1)^2$, and therefore $n' - \frac{1}{6}n(n-1)$ than $\frac{1}{6}(n-1)(2n-3)$, which is positive (except when the space to be bent is a line); and in this case the other quantities $Z_h(ij)$ are incompletely determined also.

A better way, no doubt, of stating the results (A), (B), (C) is that the functions

$$\Sigma_h [Z_h(ik)\, Z_h(jl) - Z_h(il)\, Z_h(jk)],$$
$$\Sigma_h [Z_h(ik)\, Z_h(ij) - Z_h(ii)\, Z_h(jk)],$$
$$\Sigma_h [Z_h(ii)\, Z_h(jj) - Z_h(ij)^2],$$

are invariants of the quadrics $z_1, z_2, \ldots$, with respect to the mode of transformation (not linear) $x_i + \Delta x_i = \xi_i$, determined by the condition that δs^2 shall be a covariant of the same quadrics. And I presume that the results might be obtained by a method more appropriate to this way of stating them.

If $n=2$, and $n'=1$, and we write for convenience $z = ax^2 + 2bxy + cz^2$, there is only one invariant, $ca - b^2$, giving Gauss's well-known condition.

If $n=3$, and $n'=1$, and we write

$$w = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy,$$

there are six invariants, $bc - a'^2$, $ca - b'^2$, $ab - c'^2$, and $b'c' - aa'$, $c'a' - bb'$, $a'b' - cc'$, and the space is in general completely determined. *In general*, because when the discriminant $abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2$ vanishes, the case is different. For, denoting these six quantities by A, B, C, A', B', C' for the given space, and by L, M, N, L', M', N' for the trans-

formed, we shall find that, if the discriminant vanishes for the given space, it is enongh to suppose, for instance, that $M, N, L' = B, C, A'$, and it will follow that $M', N', L = B', C', A$, so that the invariance imposes only three conditions instead of six upon the transformed system. For with the discriminant vanish

$$BC-A'^2,\quad CA-B'^2,\quad AB-C'^2,\quad B'C'-AA',\quad C'A'-BB',\quad A'B'-CC';$$

of which $BC-A'^2 = MN-L'^2$; this, therefore, vanishes also, and with it the discriminant of the transformed system, and

$$NL-M'^2,\quad LM-N'^2,\quad M'N'-LL',\quad N'L'-MM',\quad L'M'-NN.$$

In these substitute B, C, A' for M, N, L', and it is easily found that $M', N', L = B', C', A$.

It follows that plane space of three dimensions, that is, the space of actual experience, might be bent in plane space of four dimensions into forms analogous to developable surfaces; but that, in general, space of three dimensions would not admit of pure flexure in space of four dimensions.

Upon the conditions of flexure depends the possibility of applying a space to another which differs from it otherwise than merely in position; so therefore does the nature of a space of which any part may be applied to any other part indifferently in all directions, or, in other words, such that figures may move freely in it without extension or compression.

If n' is not greater than $\frac{1}{6}n(n-1)$, this is altogether impossible unless it is possible without flexure, that is, when the space of n dimensions everywhere alike in all but position. For instance, when $n = 3$ and $n' = 1$, it is only possible with plane space and with space for which, irrespectively of T, equation (1) is the same for all origins, and is of the form $2z = Z\Sigma x^2$, or, in a more familiar notation, $2Rw = x^2+y^2+z^2$. This is the mode of space shortly described in Riemann's *Habilitationsschrift*, p. 16 (III., § 2), whose geodetic surface has all the properties of the sphere (of radius R) depending only on measures or situation estimated in the surface itself. Any other mode of space of three dimensions, which should allow the free motion of figures without extension or compression, and in particular that defined by Lobatschewsky, would be impossible in plane space of four dimensions.

[Mr. Cayley's final expression for the variation of x may be verified immediately in the general form $\Delta x_i = \frac{1}{24}\frac{d\Omega}{dx_i}$, where $\Omega = -4\Sigma_h\Delta_a(z^2)$; for three differentiations in the manner of p. 171 will give $3jkl, i+ij, kl+ik, lj+il, jk = 0$, which is obviously true. The constants which would be introduced by reintegration must vanish that Δx_i may be of the third order at lowest.]

The Calculus of Equivalent Statements (Second Paper).

By Hugh McColl, B.A.

[*Read June 13th*, 1878.]

The following additions to my former article on this subject (see Vol. IX., pp. 9—20), though perhaps belonging more strictly to the province of logic than to that of mathematics, will, I hope, be found interesting.

Def. 12.—The symbol $A : B$ (which may be called an *implication*) asserts that the statement A implies B; or that whenever A is true B is also true.

Note.—It is evident that the implication $A : B$ and the equation $A = AB$ are equivalent statements. (See Rule 2.)

Rule 9.—As an extension of Rule 3 we may give the following:—

$$(ABCD\ldots)' = A' + B' + C' + D' + \ldots$$
$$= A' + AB' + ABC' + ABCD' + \ldots,$$

there being as many terms on the right-hand side of this equation as there are factors on the left-hand side.

Note.—It will be observed—(1) that the terms of the second equivalent for $(ABCD\ldots)$, are mutually exclusive; (2) that the r^{th} term of this equivalent contains r factors; and (3) that the *last factor only* of each term is accented.

Rule 10.—As an extension of Rule 5 we may give the following:

$$A + B + C + D + \ldots = A + A'B + A'B'C + A'B'C'D + \ldots;$$

there being as many terms on the right-hand side as on the left.

Note.—It will be observed—(1) that the terms on the right-hand side are mutually exclusive; (2) that the r^{th} term contains r factors; and (3) that in every term *all the factors except the last* are accented. It will also be observed that the equivalents (with mutually exclusive terms) for $(ABCD\ldots)'$ and $A + B + C + D + \ldots$ are so related that we can obtain either from the other by simply interchanging A and A', B and B', C and C', D and D', and so on throughout.

Rule 11.—If $A : B$, then $B' : A'$. Thus the implications $A : B$ and $B' : A'$ are equivalent, each following as a necessary consequence of the other. This is the logical principle of "contraposition."

Note.—The implication $A : BCD\ldots$ gives us as many simple implications as there are *factors* on the *right-hand side*, namely, the implications $A : B$, $A : C$, $A : D$, &c. The implication $A + B + C + \ldots : M$ gives us as many simple implications as there are *terms* on the *left-hand side*, namely, the implications $A : M$, $B : M$, $C : M$, &c. The equation $A = B$ gives us the two implications $A : B$ and $B : A$, from which again we get by contraposition $B' : A'$ and $A' : B'$. The implication $A : B$ is *not*

of course equivalent to $B : A$, but to $B' : A'$; for, when we transpose the statements, we must change their signs, *i.e.*, affix or remove the accent of negation. The implication $A : 0$ asserts that A is false; but $A : 1$ does not assert that A is true.

Rule 12.—If $A : B$, then $AC : BC$, whatever the statement C may be.

Rule 13.—If $A : \alpha$, $B : \beta$, $C : \gamma$, then $ABC : \alpha\beta\gamma$, and so on for any number of implications.

Rule 14.—If $AB = 0$, then $A : B'$ and $B : A'$. More generally, if $ABCD\ldots = 0$; then $A : (BCD\ldots)'$, $B : (ACD\ldots)'$, and so on; also $AB : (CD)'$, $AC : (BD)'$, and so on. In other words, when a compound statement is false, if one or more of its factors be true, one at least of the remaining factors is false.

I quote the following from Bain's "Deductive Logic":—

"The author (*i.e.* Professor Boole) extends his analysis so as to comprise a more difficult order of examples, typified thus: Suppose the analysis of a particular class of substances has conducted us to the following general conclusions, namely:—

"First, Wherever the properties A and B are combined, either the property C or the property D is present also; but they are not present jointly.

"Secondly, Wherever B and C are combined, A and D are either both present or both absent.

"Thirdly, Wherever A and B are both absent, C and D are both absent also; and *vice versâ*, where C and D are both absent, A and B are both absent also.

"Let it then be required from these conditions to determine what may be concluded in any particular instance from the presence of the property A with respect to the presence or absence of the properties B and C, paying no regard to the property D.

"The working of the corresponding equations leads to the answer:—Wherever A is present, there either C is present and B absent, or C is absent. And, inversely, wherever C is present and B is absent, there A is present."

Professor Bain does not give Professor Boole's solution of this problem, and I have not yet seen the latter's celebrated works, "The Mathematical Analysis of Logic," and "The Laws of Thought," but the problem may be easily solved by the preceding rules as follows:—

Let A denote the statement that the property A is present, and let A' denote the statement that the property A is absent. Let similar interpretations be given to B, B', C, C', D, D'.

The first two given implications are

$$AB : (C+D)\,(CD)' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$BC : AD + A'D' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and from these data alone we can find implications which contain neither D nor D', thus:

$$(C+D)\,(CD)' = (C+D)\,(C'+D') = CD' + C'D,$$

omitting the zero-terms CC' and DD'.

Hence (1) may be written $AB : CD' + C'D$.

Multiplying both members of this implication by C, we get (see rule 12), $ABC : CD'$ .....(4).

From (2) we get in the same way

$$ABC : AD \text{ .....(5).}$$

From (4) and (5) we get, by rule 13,

$$ABC : CD'AD : 0.$$

Hence $$ABC = 0.$$

That is, $A : (BC)'$ or $B'C+C'$ (see rules 14 and 3).

This is the first conclusion required, and it follows from the first two data alone, independently of the third.

The third premise is $A'B' = C'D'$.....(3),

and from this we get $A'B'C = CC'D' = 0$.

Hence $B'C : A$ (from rule 14).

This is the second conclusion required, and it follows from the third premise alone, independently of the first two.

Note.—In my quotation from Prof. Bain's work (second edition), I have ventured to correct what I believe to be two misprints; I have substituted B for D in the conclusion of the paragraph beginning "Thirdly," and I have substituted B for A in the conclusion of the final paragraph.

With reference to the preceding note and solution, the referee who reported on this paper makes the following remarks :—

"These corrections agree with the problem in Boole's 'Laws of Thought,' p. 118.

"Boole is only giving some of the conclusions from the premises, not all, nor does he imply that the conclusions are all which are deducible from the premises; on page 129 he has three other conclusions:

"(1) Wherever the property C is found, either the property A or the property B will be found with it, but not both of them together.

"(2) If the property B is absent, either A and C will be jointly present, or C will be absent.

"(3) Conversely, if A and C are jointly present, B will be absent."

From the equations $ABC = 0$ and $A'B'C = 0$, which have been already deduced from the premises, it is evident that several conclusions may be drawn by mere inspection (see rule 14). Thus from the equation $ABC = 0$ we get

$$A : (BC)', \quad B : (AC)', \quad C : (AB)',$$

$$AB : C', \quad AC : B', \quad BC : A';$$

and from the equation $A'B'C = 0$ we get

$$A' : (B'C)', \quad B' : (A'C)', \quad C : (A'B'),$$

$$A'B' : C', \quad A'C : B, \quad B'C : A.$$

From the two implications $C : (AB)'$ and $C : (A'B')'$, we get $C : (AB)'(A'B')'$, that is, $C : AB' + A'B$, as may be seen immediately by writing suitable equivalents for $(AB)'$ and $(A'B')'$ and multiplying. This is the *first* of the additional conclusions. The *second* is the implication $B : (A'C)'$, and the *third* is the implication $AC : B'$.

Note.—Prof. Boole's third premise is $A'B' = C'D'$, which is equivalent to two implications, namely, $A'B' : C'D'$ and $C'D' : A'B'$. The first of these implications is the only one I have used; possibly additional conclusions might be obtained by the help of the second implication, but at present I cannot see how.

This paper, as originally sent to the Mathematical Society, contained an application of the preceding principles and notation to the Syllogism. The gentleman (already referred to) who reported on my paper, very justly objected that, like Prof. Boole, I introduced a new and quite unnecessary term into the syllogism. I now propose, instead of the method thus objected to, to substitute the following modification of it, which is not open to the same objection.*

Def. 13.—The symbol $A \div B$ asserts that A does *not* imply B; it is thus equivalent to the less convenient symbol $(A : B)'$.

Note.—The symbol $A \div B$ thus asserts that the truth of B is not a *necessary* consequence of the truth of A; in other words, it asserts that the statement A is consistent with B', but it makes no assertion as to whether A is consistent with B or not.

Rule 15.—If A implies B and B implies C, then A implies C.

Rule 16.—If A does not imply B, then B' does not imply A'; in other words, the non-implications $A \div B$ and $B' \div A'$ are equivalent.

Note.—This is easily proved as follows: $A \div B = (A : B)' = (B' : A')' = B' \div A'$. Thus in a non-implication, as in an implication, the rule is *Transpose and change signs* (see Rule 11).

Rule 17.—If A implies B but does not imply C, then B does not imply C; in other words, from the two premises $A : B$ and $A \div C$, we get the conclusion $B \div C$.

Rule 18.—If A (assuming it to be a consistent statement) implies B, then A does not imply B'; in other words, from the implication $A : B$ we deduce the non-implication $A \div B'$, which is equivalent to $B \div A'$.

Note.—The non-implication $AB \div 0$ is the same as $A \div B'$.

Rules 15, 17, 18 combined with the principle of contraposition, expressed in Rules 11 and 16, will be found to include all the valid syllogisms, to the examination of which we now proceed.

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* See the geometrical illustrations on p. 182, 183.

The statement *All X is Y* may be denoted by the implication $x : y$, in which x denotes the statement that a certain representative individual belongs to the class X, and y denotes the statement that he belongs to the class Y. Similar interpretations may be put upon z, u, v, α, β, &c. The number of individuals belonging to any class may be known or unknown, constant or varying, finite or infinite.

Note.—There is some analogy between this analytical kind of logic and analytical geometry; for, just as in the former the statements x, y, z, &c. refer to *a representative individual of a class*, so in the latter the distances x, y, z, &c. have reference to *a representative point in a locus.*

The statement *No X is Y* will thus be expressed by the implication $x : y'$, which, on the principle of contraposition, is equivalent to the implication $y : x'$ (see Rule 11).

The statement *Some X is Y* will be expressed by the non-implication $x \div y'$, which, by Rule 16, is equivalent to the implication $y \div x'$. *Some* means *one individual at least.*

The statement *Some X is not Y* will be expressed by the non-implication $x \div y$, which, by Rule 16, is equivalent to the implication $y' \div x'$.

These conventions being laid down, the ordinary nineteen syllogisms are as follows, the order for each syllogism (read downwards) being *Major Premise, Minor Premise, Conclusion :—*

First Figure.

$y : z$	$y : z$	$y : z$	$y : z$
$x : y$	$x : y$	$x \div y'$	$x \div y'$
$x : z$	$x : z'$	$x \div z'$	$x \div z$

Note.—The first two moods in this figure are direct applications of Rule 15, and the other two are applications of Rule 17 combined with the principle of contraposition. Thus, by contraposition, the second premise in the third mood may be written $y \div x'$, and from the two premises we then get the conclusion $z \div x'$, which, by contraposition, is equivalent to $x \div z'$.

Second Figure.

$z : y'$	$z : y$	$z : y'$	$z : y$
$x : y$	$x : y'$	$x \div y'$	$x \div y$
$x : z'$	$x : z'$	$x \div z$	$x \div z$

Note.—As in the preceding figure, each mood may be brought under Rule 15 or 17 by transposing one or more of the premises, and making the necessary alteration in the accent.

Third Figure.

$y : z$	$y \div z'$	$y : z$	$y : z'$	$y \div z$	$y : z'$
$y : x$	$y : x$	$y \div x'$	$y : x$	$y : x$	$y \div x'$
$x \div z'$	$x \div z'$	$x \div z'$	$x \div z$	$x \div z$	$x \div z$

Note.—The second premise in the first mood of this figure is $y : x$; from the first premise we deduce the non-implication $y \div z'$ by Rule 18, and then the conclusion

follows from Rule 17. The conclusion in the second mood follows directly from the premises by Rule 17. In the third mood we get $z \div x'$ from the premises by Rule 17, and this by contraposition (see Rule 16) is equivalent to $x \div z'$. The remaining three moods in like manner come under Rule 17 either directly or by contraposition, or by the help of Rule 18.

FOURTH FIGURE.

$z : y$	$z : y$	$z \div y'$	$z : y'$	$z : y'$
$y : x$	$y : x'$	$y : x$	$y : x$	$y \div x'$
$x \div z'$	$x : z'$	$x \div z'$	$x \div z$	$x \div z$

Note.—In the first mood we use Rules 15 and 18. In the second mood we use Rule 15 and contraposition. In the third mood we use contraposition and Rule 17. In the fourth mood we proceed as follows:—The first premise is equivalent to $y : z'$ by contraposition, the second premise gives the non-implication $y \div x'$ by Rule 18, the conclusion $z' \div x'$ follows from Rule 17, and this by contraposition is equivalent to $x \div z$. In the fifth mood $y : z'$ is an equivalent for the first premise; from this and the second premise we get the conclusion $z' \div x'$ by Rule 17, and this conclusion by contraposition is equivalent to $x \div z$.

Fresh syllogisms may be obtained by changing the sign of x throughout—*i. e.*, by affixing or removing the accent of negation; but these nineteen fresh syllogisms, like the nineteen ordinary ones already given, will all come under Rule 15 or 17, either directly or by the principle of contraposition, or by the aid of Rule 18.

The cardinal difference between my present method of treating the syllogism and my former one consists in the introduction of the symbol $\div$ as the denial of $:$, so that the statements $A : B$ and $A \div B$ stand in precisely the same relation to each other as α and α'. In fact, if we denote either statement by α, we must denote the other by α'. In my former method I expressed the statement *Some X is Y* by the implication $v : xy$, in which v denoted the statement that the representative individual spoken of belonged to some class V common both to the class X and the class Y. This was the point objected to in the report on my former paper.

GEOMETRICAL ILLUSTRATIONS.

The meanings of the symbols $x : y$, $x : y'$, $x \div y$, $x \div y'$, $x = y$ may be illustrated geometrically as follows:

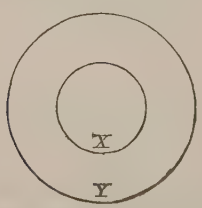

FIG. 1. $(x : y)\,(y \div x)$.

FIG. 2. $x : y'$.

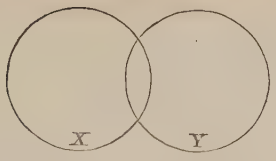

FIG. 3. $(x \div y)\,(x \div y')$.

FIG. 4. $x = y$.

Let x denote the statement that a representative point is within the boundary X, and let y denote the statement that it is within the boundary Y. Then Fig. 1 illustrates the compound statement $(x : y)\,(y \div x)$, the first factor of which is an implication, the second a non-implication; $x : y'$ is illustrated by Fig. 2; Fig. 3 illustrates the product of the two non-implications $x \div y$ and $x \div y'$; and Fig. 4 illustrates $x = y$, the boundary X coinciding with the boundary Y.

The various syllogisms may be similarly illustrated. For instance, the first syllogism may be illustrated by Fig. 5.

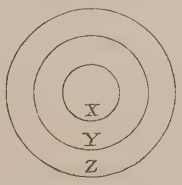

FIG. 5. $x : y : z$.

APPENDIX.

Since $\alpha : \beta$ asserts that β is a factor of α, the implication $\alpha : 0$ asserts that α is false, and the implication $1 : \alpha$ asserts that α is true; but the implications $0 : \alpha$ and $\alpha : 1$ make no assertion as to whether α is true or false. In other words, a statement which contains a false factor must be false, and a statement which is a factor of a true statement must be true; but a factor of a false statement may be true, and a statement which contains a true factor may nevertheless be false.

Every possible valid syllogism comes under one or other of the four following standard implications, either directly or by the substitution of equivalents:

$$(\alpha : \beta)\,(\beta : \gamma) : (\alpha : \gamma) \quad \ldots\ldots\ldots\

the same axiom, since $(\alpha : \gamma)$ implies $(\alpha \div \gamma')$;* and the third and fourth may be deduced from the first as follows:

Multiplying both sides of the first implication by $(\alpha \div \gamma)$, we get

$$(\alpha : \beta)\ (\beta : \gamma)\ (\alpha \div \gamma) : 0\ ;$$

and the third implication follows from this by Rule 14. In the second implication, change γ into γ', and therefore γ' into γ, and it becomes

$$(\alpha : \beta)\ (\beta : \gamma') : (\alpha \div \gamma).$$

Multiplying both sides of this implication by $(\alpha : \gamma)$, we get

$$(\alpha : \beta)\ (\beta : \gamma')\ (\alpha : \gamma) : 0\ ;$$

and the fourth implication follows from this by Rule 14.

Let $f(\alpha, \beta, \gamma)$, $\phi(\alpha, \beta, \gamma)$, $\psi(\alpha, \beta, \gamma)$, $\chi(\alpha, \beta, \gamma)$ denote the foregoing four standard implications respectively. It is easily seen that $f(\alpha, \beta, \gamma) = f(\gamma', \beta', \alpha')$, and that $\chi(\alpha, \beta, \gamma) = \chi(\alpha, \gamma, \beta)$. Considering all those syllogisms equivalent which have equivalent premises and the same or equivalent conclusions, the ordinary nineteen syllogisms are the following:—

Fig. 1.

$$f(x, y, z),\quad f(x, y, z'),\quad \psi(y, z, x'),\quad \psi(y, z', x').$$

Fig. 2.

$$f(x, y, z'),\quad f(x, y', z'),\quad \psi(y, z', x'),\quad \psi(y', z', x').$$

Fig. 3.

$$\chi(y, x, z),\quad \psi(y, x, z'),\quad \psi(y, z, x'),\quad \chi(y, x, z'),\quad \psi(y, x, z),\quad \psi(y, z', x').$$

Fig. 4.

$$\phi(z, y, x),\quad f(x, y', z'),\quad \psi(y, x, z'),\quad \chi(y, x, z'),\quad \psi(y, z', x').$$

If we number these syllogisms 1, 2, 3, ... 18, 19, we may see that $2=5$, $3=11$, $4=7=14=19$, $6=16$, $10=17$, and $12=18$. Eight of the nineteen syllogisms may therefore be looked upon as redundant. On the other hand, several valid syllogisms which are not found among the nineteen might be introduced by admitting the implications and non-implications $y' : z$, $y' \div z$, $y' : x$, $y' \div x$ (or their equivalents $z' : y$, $z' \div y$, &c.) as factors in the premises, which would lead to similar expressions in x and z in the conclusion.

Note.—The implication $\alpha : \beta'$ asserts that α and β are inconsistent with each other; the non-implication $\alpha \div \beta'$ asserts that α and β are consistent with each other.

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* This assumes that α is a *consistent* statement—*i. e.*, one which *may* be true.

Supplement.

Since the preceding article was put in type I have discovered the following method of elimination, under which Prof. Boole's problem may be brought as a particular case:

Let $f(x)$ denote any statement involving x; and let $f'(x)$ denote the denial of this statement, so that $f'(x)$ is equivalent to $\{f(x)\}'$. Let $\phi(x)$ and $\phi'(x)$ be similarly interpreted. Suppose now that the implication $f(x) : \phi(x)$ is given, and that we have to solve this implication with regard to x. The required solution is

$$f(0)\,\phi'(0) : x, \quad f(1)\,\phi'(1) : x'.$$

Let the antecedents of x and x' in this result be respectively denoted by α and β; that is, let α denote $f(0)\,\phi'(0)$, and let β denote $f(1)\,\phi'(1)$. Then we have

$$\alpha : x, \quad \beta : x'.$$

Suppose next that, instead of the single given implication $f(x) : \phi(x)$, we have a *series* of such implications, and that from these given implications we get the solutions

$$\alpha_1 : x, \quad \alpha_2 : x, \quad \alpha_3 : x, \quad \&c.;$$
$$\beta_1 : x', \quad \beta_2 : x', \quad \beta_3 : x', \quad \&c.;$$

that is,

$$\alpha_1+\alpha_2+\alpha_3+\ldots : x,$$
$$\beta_1+\beta_2+\beta_3+\ldots : x'.$$

Then evidently

$$(\alpha_1+\alpha_2+\alpha_3+\ldots)\,(\beta_1+\beta_2+\beta_3+\ldots) : 0.$$

When we have performed the multiplication, we shall have a series of implications, each of the form $\alpha_m\beta_n : 0$, and none of these will involve x or x', so that we have eliminated x.

Note.—Zero terms may, of course, be omitted. Any implication of the form $0 : x$ conveys no information, and may therefore be omitted. If the final implication be of the form $0 : xx'$, that is, $0 : 0$, we may infer that the given implications are not sufficient for the elimination of x.

The *zero terms* here referred to are terms which are zero *independently of the given implications of the problem;* such, for instance, as $AA'B$, $BC \times 0$, &c. Thus an implication of the form $AA'B : x$ or $BC \times 0 : x'$ may be omitted; but we must not omit an implication of the form $AB : x$, unless, *independently of the given implications of the problem,* we know A and B to be inconsistent.

Since, as is evident from Rule 5, $\alpha = \alpha+\alpha = \alpha+\alpha+\alpha = \&c.$, and, more generally, $\alpha = \alpha+\alpha\beta = \alpha+\alpha\beta+\alpha\gamma = \&c.$, we may omit, in any

indeterminate expression, any terms that are *multiples* of others. For example, in the expression $\alpha\beta+\underline{\gamma\delta}+\gamma+\underline{\alpha\beta\delta}+\underline{\alpha\beta}$, we may omit the terms underlined, so that the expression reduces to $\alpha\beta+\gamma$.

In Prof. Boole's problem, putting x for D, and therefore x' for D', the given implications are,

$$\begin{aligned} AB &: C'x+Cx', \\ BC &: Ax+A'x', \\ A'B' &: C'x', \\ C'x' &: A'B'. \end{aligned}$$

Putting, as before, a_1, a_2, a_3, a_4 for the four antecedents of x, and putting β_1, β_2, β_3, β_4 for the four antecedents of x', we get

$$a_1 = ABC', \quad a_2 = BCA, \quad a_3 = A'B'C, \quad a_4 = C'(A+B);$$
$$\beta_1 = ABC, \quad \beta_2 = BCA', \quad \beta_3 = A'B', \quad \beta_4 = 0;$$

and it will be found upon trial that

$$(a_1+a_2+a_3+a_4)(\beta_1+\beta_2+\beta_3) = (AB+A'B')C;$$

so that the complete solution of the problem is furnished by the equation

$$(AB+A'B')C = 0.$$

As another example, let it be required to eliminate C (instead of D) from the data of Prof. Boole's problem. Putting x for C, the given implications are

$$\begin{aligned} AB &: x'D+xD', \\ Bx &: AD+A'D', \\ A'B' &: x'D', \\ x'D' &: A'B'. \end{aligned}$$

Interpreting a_1, a_2, &c., β_1, β_2, &c. as before, we get

$$a_1 = ABD', \quad a_2 = 0, \quad a_3 = A'B'D, \quad a_4 = D'(A+B)$$
$$\beta_1 = ABD, \quad \beta_2 = B(A'D+AD'), \quad \beta_3 = A'B', \quad \beta_4 = 0.$$

Hence $(a_1+a_2+a_3+a_4)(\beta_1+\beta_2+\beta_3+\beta_4) = ABD'+A'B'D;$

and the complete solution is

$$ABD'+A'B'D = 0.$$

Similarly, if we eliminate B, we get

$$AC'D'+A'CD = 0,$$

and if we eliminate A, we get

$$B(CD+C'D') = 0.$$

On the Decomposition of Certain Numbers into Sums of Two Square Integers by Continued Fractions. By Samuel Roberts.

[*Read June 13th*, 1878.]

1. I do not propose to enter upon the theory of these numbers with any completeness. A great number of theorems exist relating to them, though it has not been thought worth while to work out their special theory apart from more general speculations. Nevertheless, some remarks will be found in the addition to Professor Smith's "Note on the Theory of the Pellian Equation and of Binary Quadratic Forms," (*Proceedings*, Vol. VII., pp. 204–208), which shew, as it seems to me, that results of interest might be expected from the minute study of these forms. Of course, a great deal of their theory must consist of particular applications of known general propositions; and if we know the prime factors of a number, we can directly obtain all possible decompositions of it into sums of two square integers, because any prime factor of the form $4m+1$ can be so decomposed by continued fractions or an equivalent process in the theory of forms. There are, however, many relations among the different decompositions which do not appear *primâ facie*. The numbers of this kind most intimately connected with continued fractions are those numbers D which render the indeterminate equation $t^2-Du^2=-1$ resoluble in integers, and it is with these that this paper is chiefly occupied.

2. Let D be a positive odd integer containing no square factors, and such that $t^2-Du^2=-1$ is resoluble in integers. The decompositions of D into sums of two square integers are systematically obtained by developing into continued fractions the positive roots of a set of forms of the type $-Mx^2+Ny^2$, where M, N are positive conjugate factors of D and M is $< \sqrt{D}$.

For if $\sqrt{\frac{N}{M}}$ be expressed by the continued fraction

$$a+\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_2}+\frac{1}{a_1}+\frac{1}{2a}+\ldots \quad \&c.,$$

there will be found two reduced forms of the types $(P, Q, -P)$, $(-P, Q, P)$ in the period of forms derived from the fraction, provided $Mt^2-Nu^2=-M$ is resoluble in integers, which implies that the period of the fraction is odd. But this equation will be resoluble if $t^2-Du^2=-1$ is so. One of the reduced forms in question will be obtained from the periods of odd rank, and the other from the periods of even rank in

the fraction. Let μ be the number of prime factors of D, which factors will necessarily be of the form $4m+1$.

The forms $(-M, O, N)$ determine and severally belong to the $2^{\mu-1}$ properly primitive ambiguous classes of Determinant D,* and therefore each of the decompositions thus obtained will be distinct from the rest. Hence, because only $2^{\mu-1}$ such decompositions exist, we shall have obtained them all when we have developed the series of fractions beginning with a complete quotient of the type $\frac{\sqrt{D}+0}{M}$.

We may equally make use of the improperly primitive ambiguous forms $\left(2M, M, -\frac{N-M}{2}\right)$, taking $M<N$ as before, and developing the fractions whose first complete quotients are of the form $\frac{\sqrt{D}+M}{2M}$. And, generally, we may take other forms, provided an ambiguous class corresponding to each factor M is represented. Thus we may set out with the set of forms $(-M, M, N-M)$, or the set

$$\{(m+1)^2 M-m^2 N,\ (m+1)M-mN,\ M-N\},$$

where m is an arbitrary integer. Such sets as

$$\left(-1,\ M+N,\ -\frac{M^3-N^3}{M-N}\right) \text{ or } \{-1, M, M(N-M)\}$$

all belong to the principal class, and will only give what may be termed the principal decomposition, because it depends on the factor unity.

3. If D is even but without square factors, the result is similar, except that there are no improperly primitive ambiguous forms. We may take for M any odd integer factor less than $\sqrt{\frac{D}{2}}$. The doubles of these factors will give the decompositions over again.

When D contains square factors consisting, as will be the case, of products of primes of the form $4m+1$, the process will still be the same. Relatively prime factors M, N will give the decompositions into relatively prime squares. When M, N are not relatively prime, the resulting decompositions will contain the common factor squared.

If D does not render $t^2-Du^2=-1$ resoluble by reason of its containing a factor σ^2 consisting of products of primes of the form $4m+3$ or of a power of 2 or both combined, or generally a factor σ^2 not contained in any number u^2 belonging to a solution of $t^2-\frac{D}{\sigma^2}u^2=-1$, yet $\sigma t^2-\frac{D}{\sigma}u^2=-\sigma$ may be resoluble, and will be so if $t^2-\frac{D}{\sigma^2}u^2=-1$ is so. We can in this case proceed as before, only taking factors M, N

* Gauss, Disq. Arith., pp. 257, 258.

having σ for a common factor. There are, therefore, cases in which we thus obtain some, not all, of the decompositions.

The decompositions will depend, in a definite manner, on particular factors of D; and the same thing appears, though less distinctly, in the direct decompositions by means of complex factors or otherwise.

Example: $D = 3^2 . 2^3 . 5 . 13 = 4680$. I write I for "greatest integer in." The decomposition depending on the factor $3 . 2 . 5$ is given by

$$I\frac{\sqrt{D}+0}{30} = 2, \quad 30 . 2 - 0 = 60, \quad \frac{D-60^2}{30} = 36,$$

$$I\frac{\sqrt{D}+60}{36} = 3, \quad 36 . 3 - 60 = 48, \quad \frac{D-48^2}{36} = 66,$$

$$I\frac{\sqrt{D}+48}{66} = 1, \quad 66 . 1 - 48 = 18, \quad \frac{D-18^2}{66} = 66,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$D = 18^2 + 66^2.$$

Of course it is practically simpler to take out the square factor at once and restore it subsequently.

4. Beginning with the complete quotient $\frac{\sqrt{D}+B}{A} = \omega$, a positive root greater than unity of

$$-A\omega^2 + 2B\omega + C = 0,$$

we have the series of equalities (according to a notation which explains itself) $\frac{\sqrt{D}+B_p}{A_p} = a_p, \quad A_p a_p - B_p = B_{p+1}, \quad \frac{D-B_{p+1}^2}{A_p} = A_{p+1},$

$$A_{p+1} = -A_p a_p^2 + 2B_p a_p + A_{p-1};$$

and if $\frac{u_p}{v_p}$ is the convergent $\frac{(a, \ a_p)}{(a_1, \ a_p)}$,

$$A_{p+1} = (-)^p \{-Au_p^2 + 2Bu_p v_p + Cv_p^2\}.$$

Also, if we put $\frac{\sqrt{D}+B_p}{A_p} = (-)^p \frac{x}{y}$, we get

$$\phi_{p+1} = (-)^{p+1} A_p x^2 + 2B_p xy \, (-)^{p+2} A_{p-1} y^2.$$

Hence ϕ_p is changed into the contiguous form (by the left) ϕ_{p+1} by the proper substitution $\begin{Bmatrix} (-)^{p-1} a_{p-1}, & -1 \\ 1, & 0 \end{Bmatrix} = 1.$

If the form ϕ_{p+2} is reduced and of the type $(a, \ b, \ -a)$, we must have $A_p = A_{p+1}$, and the condition is

$$-A(u_p^2 + u_{p-1}^2) + 2B(u_p v_p + u_{p-1} v_{p-1}) + C(v_p^2 + v_{p-1}^2) = 0.$$

In fact, if we assume

$$a(\alpha^2 + \gamma^2) + 2b(\alpha\beta + \gamma\delta) + c(\beta^2 + \delta^2) = 0,$$

the following expression is an identity

$$(a\delta-\beta\gamma)^2\,(b^2-ac) = [a\alpha\gamma+b\,(a\delta+\beta\gamma)+c\beta\delta]^2+(a\alpha^2+2ba\beta+c\beta^2)^2,$$

or

$$= [a\alpha\gamma+b\,(a\delta+\beta\gamma)+c\beta\delta]^2+(a\gamma^2+2b\gamma\delta+c\delta^2)^2\,;$$

and the conditions are equivalent to

$$a\nu-2b\mu+c\lambda=0,$$

$$\mu^2-\lambda\nu=-1,$$

$a\delta-\beta\gamma=\pm 1$ being assumed. (*Proceedings*, Vol. VII., p. 206.) For any given consecutive convergents $\frac{\alpha}{\beta}$, $\frac{\gamma}{\delta}$, $\alpha>\beta$, $\gamma>\delta$, $\beta>\delta$,* we can determine an infinite number of values a, b, c, and each set determines a number decomposable into the sum of two squares. Among these solutions will always be found some which give forms belonging to an ambiguous class. In these cases $t^2-du^2=-1$ will be resoluble if a, b, c contain no common factor, or $t^2-\frac{D}{\sigma^2}u^2=-1$ if they contain a factor σ.

5. Since the form ϕ_{p+2} is obtained from the form

$$\phi_0=-Ax^2+2Bxy+Cy^2$$

by the composite substitution

$$\begin{pmatrix}a, & -1\\ 1, & 0\end{pmatrix}\begin{pmatrix}-a_1, & -1\\ 1, & 0\end{pmatrix}\begin{pmatrix}a_2, & -1\\ 1, & 0\end{pmatrix}\cdots\cdots\begin{pmatrix}\pm a_p, & -1\\ 1, & 0\end{pmatrix}=\begin{Bmatrix}\alpha, & \mp\gamma\\ \beta, & \mp\delta\end{Bmatrix}$$

according as p is even or odd, we shall obtain the form $(-A, B, C)$ from the form $(\pm P, Q, \mp P)$ by the inverse substitution $\begin{Bmatrix}\mp\delta, & \pm\gamma\\ -\beta, & \alpha\end{Bmatrix}$ according as p is even or odd, or according as $a\delta-\beta\gamma$ is -1 or $+1$.

We have therefore, in the two cases,

$$\pm\,[P\,(\delta^2-\beta^2)+2Q\beta\delta]=-A, \qquad \pm\,[P\,(\gamma^2-\alpha^2)+2Q\alpha\gamma]=C,$$

$$\pm\,[P\,(\alpha\beta-\gamma\delta)-Q\,(\alpha\delta+\beta\gamma)]=B\,;$$

and the determinant

$$\begin{Bmatrix}-A, & B, & C\\ 2\beta\delta, & -(a\delta+\beta\gamma), & 2a\gamma\\ \delta^2-\beta^2, & a\beta-\gamma\delta, & \gamma^2-a^2\end{Bmatrix}=\mp\{-A\,(\alpha^2+\gamma^2)+2B\,(\alpha\beta+\gamma\delta)+C\,(\beta^2+\delta^2)\}=0$$

gives the same condition as before.

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* In certain limiting cases, as, *e.g.*, when the period of the fraction consists of one term, provisional adaptations are necessary in order to preserve the continuity of notation. This is the case generally in the theory of continued fractions. Thus it is often convenient to take to $\frac{1}{0}$ as a first convergent.

6. If $A = M$, $B = 0$, $C = N$, the condition becomes $\frac{N}{M} = \frac{\alpha^2+\gamma^2}{\beta^2+\delta^2}$, and $\beta^2+\delta^2$ must contain M if M is prime to N.

Let $R^2+S^2 = \frac{N}{M}$; then

$$(S+iR)(\beta-i\delta) = \alpha+i\gamma$$

gives $$R = \frac{\alpha\delta+\beta\gamma}{\beta^2+\delta^2}, \quad S = \frac{\alpha\beta-\gamma\delta}{\beta^2+\delta^2};$$

and if $MN = D = P^2+Q^2$, we may put

$$P = \frac{M(\alpha\delta+\beta\gamma)}{\beta^2+\delta^2}, \quad Q = \frac{M(\alpha\beta-\gamma\delta)}{\beta^2+\delta^2}.$$

Let ϵ, η denote (a_2, a_p) and (a_2, a_{p-1}), so that

$$\alpha = \kappa\beta+\epsilon, \quad \gamma = \kappa\delta+\eta, \quad \epsilon\delta-\eta\beta = \mp 1;$$

we have

$$(a.) \quad P = \frac{2\beta\delta M}{\beta^2+\delta^2}\left(\kappa+\frac{\epsilon\delta+\eta\beta}{2\beta\delta}\right),$$

$$(b.) \quad Q = \frac{\beta^2-\delta^2 . N}{\beta^2+\delta^2}\left(\kappa+\frac{\epsilon\beta-\eta\delta}{\beta^2-\delta^2}\right).$$

For the principal solution, when $M = 1$,

$$P = I\frac{2\beta\delta}{\beta^2+\delta^2}(k+1), \quad Q = I\left(\frac{\beta^2-\delta^2}{\beta^2+\delta^2}\right)(k+1),$$

or $$Q+iP = I\frac{\beta+i\delta}{\beta-i\delta}(k+1), \quad \text{where } k = I\surd D.$$

It is a somewhat more tedious matter to determine corresponding limits for decompositions depending on a factor $M >$ unity.

Since, in consequence of

$$\epsilon\delta-\eta\beta = \mp 1, \quad \text{and} \quad \beta^2+\delta^2 = M(\beta'^2+\delta'^2),$$

$$D = M^2\kappa^2+2M^2\kappa\frac{\beta\epsilon+\eta\delta}{\beta^2+\delta^2}+M^2\frac{\epsilon^2+\eta^2}{\beta^2+\delta^2} = M^2\left(\kappa+\frac{\beta\epsilon+\eta\delta}{\beta^2+\delta^2}\right)^2+\frac{1}{(\beta'^2+\delta'^2)^2}$$

$$= \left(k+\frac{\beta'\epsilon'+\eta'\delta'}{\beta'^2+\delta'^2}\right)^2+\frac{1}{(\beta'^2+\delta'^2)^2},$$

where the accents denote that the numbers belong to the case of $M = 1$ and $k = I\surd D$, we have

$$M\left(\kappa+\frac{\beta\epsilon+\eta\delta}{\beta^2+\delta^2}\right) = k+\frac{\epsilon'\beta'+\eta'\delta'}{\beta'^2+\delta'^2} \gtreqless k < k+1.$$

Hence, by (a) and (b),

$$P = \frac{2\beta\delta}{\beta^2+\delta^2}k+\frac{2\beta\delta(\epsilon'\beta'+\eta'\delta')-(\epsilon\delta-\eta\beta)(\beta^2-\delta^2)}{(\beta^2+\delta^2)(\beta'^2+\delta'^2)},$$

$$Q = \frac{\beta^2-\delta^2}{\beta^2+\delta^2}k + \frac{(\beta^2-\delta^2)(\epsilon'\beta'+\eta'\delta')+(\epsilon\delta-\eta\beta)\,2\beta\delta}{(\beta^2+\delta^2)(\beta'^2+\delta'^2)},$$

because

$$\frac{\epsilon\beta+\eta\delta}{\beta^2+\delta^2} - \frac{\epsilon\delta+\eta\beta}{2\beta\delta} = \frac{(\epsilon\delta-\eta\beta)(\beta^2-\delta^2)}{2\beta\delta(\beta^2+\delta^2)},$$

$$\frac{\epsilon\beta+\eta\delta}{\beta^2+\delta^2} - \frac{\epsilon\beta-\eta\delta}{\beta^2-\delta^2} = \frac{(\eta\beta-\epsilon\delta)(2\beta\delta)}{(\beta^2-\delta^2)(\beta^2+\delta^2)}.$$

I. $$\epsilon\delta-\eta\beta = +1, \quad P-\frac{2\beta\delta}{\beta^2+\delta^2}k > 0;$$

then, since $\epsilon'\beta'+\eta'\delta'$ is $< \beta'^2+\delta'^2$,

$$2\beta\delta(\epsilon'\beta'+\eta'\delta')-(\beta^2-\delta^2) \text{ is } < 2\beta\delta(\beta'^2+\delta'^2),$$

and according as $2\beta\delta$ is $> (\beta^2-\delta^2)$, or $(\beta^2-\delta^2) > 2\beta\delta$,

$$(\beta^2-\delta^2)(\epsilon'\beta'+\eta'\delta')+2\beta\delta \gtreqless 2\beta\delta(\beta'^2+\delta'^2) \gtreqless (\beta^2-\delta^2)(\beta'^2+\delta'^2),$$

under the same conditions.

Therefore $$P > \frac{2\beta\delta}{\beta^2+\delta^2}k < \frac{2\beta\delta}{\beta^2+\delta^2}k+1 = I\left(\frac{2\beta\delta}{\beta^2+\delta^2}k\right)+1,$$

$$Q > \frac{\beta^2-\delta^2}{\beta^2+\delta^2}k < \frac{\beta^2-\delta^2}{\beta^2+\delta^2}k+1 = I\left(\frac{\beta^2-\delta^2}{\beta^2+\delta^2}k\right)+1;$$

and we have the same inequalities of

$$\epsilon\delta-\eta\beta = -1, \quad Q-\frac{\beta^2-\delta^2}{\beta^2+\delta^2}k > 0.$$

II. If $$\epsilon\beta-\eta\delta = +1, \text{ and } P-\frac{2\beta\delta}{\beta^2+\delta^2}k < 0,$$

we have similarly

$$P < \frac{2\beta\delta}{\beta^2+\delta^2}k > \frac{2\beta\delta}{\beta^2+\delta^2}k-1 = I\frac{2\beta\delta}{\beta^2+\delta^2}k,$$

$$Q > \frac{\beta^2-\delta^2}{\beta^2+\delta^2}k < \frac{\beta^2-\delta^2}{\beta^2+\delta^2}k+1 = I\left(\frac{\beta^2-\delta^2}{\beta^2+\delta^2}k\right)+1;$$

and if $$\epsilon\beta-\eta\delta = -1, \qquad Q-\frac{\beta^2-\delta^2}{\beta^2+\delta^2}k < 0,$$

$$P > \frac{2\beta\delta}{\beta^2+\delta^2}k < \frac{2\beta\delta}{\beta^2+\delta^2}k+1 = I\left(\frac{2\beta\delta}{\beta^2+\delta^2}k\right)+1,$$

$$Q < \frac{\beta^2-\delta^2}{\beta^2+\delta^2}k > \frac{\beta^2-\delta^2}{\beta^2+\delta^2}k-1 = I\frac{\beta^2-\delta^2}{\beta^2+\delta^2}k.$$

The case II. applies when D is of the form t^2+1. For example, in the celebrated form $2^{32}+1$, the factor $641 = 25^2+4^2$. Hence

$$P-1 = I\frac{5^2}{641}2^{19} \qquad Q = I\frac{29.7.3}{641}2^{16}$$
$$= 20448 \qquad = 62264$$
$$= (11.13)^2-1; \qquad = 8.43.181.$$

We have then a fourth power and a square for the second decomposition.

The other composite form lately found,* $2^{2^{11}}+1$, which is divisible by the prime $7.2^{14}+1$ or $114689 = 260^2+217^2$, gives

$$P-1 = I\frac{2.260.217}{114689}2^{2048} \qquad Q = I\frac{43.477}{114689}2^{2048}$$
$$= I\frac{5.7.13.31}{114689}2^{2051}; \qquad = I\frac{3^2.43.53}{114689}2^{2048}.$$

The values of P and Q can evidently be exactly determined for numbers of the form t^2+k^2, when $a = p^2+q^2$, the given factor, is prime.

For then $P = \dfrac{2pqt \pm (p^2-q^2)k}{a}$, $\qquad Q = \dfrac{(p^2-q^2)t \mp 2pqk}{a}$.

7. From $D = \dfrac{M^2(\alpha^2+\gamma^2)}{\beta^2+\delta^2} = M^2\left(k^2+2k\dfrac{\beta\epsilon+\eta\delta}{\beta^2+\delta^2}+\dfrac{\epsilon^2+\eta^2}{\beta^2+\delta^2}\right)$

we must have for D integer,

$$2Mk = M(\beta\epsilon+\eta\delta)(\epsilon^2+\eta^2)+m(\beta^2+\delta^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (c);$$

and, substituting in (a) and (b), we get

$$P = M(\epsilon^3\delta+\eta^3\beta)+m\beta\delta,$$
$$2Q = M\{(\epsilon^3\beta-\eta^3\delta)+3\epsilon\eta(\eta\beta-\epsilon\delta)\}+m(\beta^2-\delta^2)$$

(*Proceedings*, Vol. VIII., p. 217).

The numbers β, δ cannot both be even on account of $\alpha\delta-\beta\gamma = \pm 1$, or both odd on account of (c), and therefore m must be even. I suppose δ not to be zero; when this is the case, D is of the form $M^2(p^2+1)$. We may, if we please, use negative values of m with suitable adaptation.†

8. If $t^2-Du^2 = -1$, and also $t^2-\sigma^2Du^2 = -1$, are resoluble, σ is a factor of U belonging to (T, U), a solution of the first equation; and *vice versâ*, if σ is a factor U, the second equation is resoluble. But

$$T_{2x+1}-\sqrt{D}\,U_{2x+1} = (t-\sqrt{D}\,u)^{2x+1};$$

and if we make $2x+1 = 4p+1 = \alpha$ any odd factor of D, U_α will be

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* "Nature," Vol. XVIII., page 104. The detection of another composite number $2^{2^{23}}+1$ is due to the same discoverer, M. Pervouchine,—"Nature," Vol. XVIII., p. 456.

† For simplicity's sake I have also supposed M odd and prime to N. The results can be adapted to the case of M, N containing a common factor.

divisible by α, and the same process may be repeated. Hence, if α, β, γ, &c. are odd factors of $D = 2^\mu \alpha^a \beta^b \gamma^c$, &c. ($\mu = 0$ or 1), and D is a number such that $t^2 - Du^2 = -1$ is resoluble, then

$$D' = 2^\mu \alpha^{a+2p} \beta^{b+2q} \gamma^{c+2r}, \text{ \&c.}$$

is also a number such that $t^2 - D'u^2 = -1$ is resoluble.

Thus, since every prime $4p+1$ is a number D, any odd power of such a prime is a number D'; and if $t^2+1 = 2^\mu \alpha^a \beta^b \gamma^c$, &c., then $2^\mu \alpha^{a+2p} \beta^{b+2q} \gamma^{c+2r}$, &c. is such a number, $\mu = 0$ or 1.

When it is required to determine whether a number D is such that $t^2 - Du^2 = -1$ is resoluble, the first step is to divide out the greatest square factor. The quotient must, as a necessary but not a sufficient condition, be a number of the same kind. If this is found to be the case, we may next multiply the quotient by the square of each prime factor which appeared originally in an even power.

The number D is thus reduced to the simpler form $2^\mu . D_1 . D_2^2$ ($\mu = 0$ or 1), D_1, D_2 being relatively prime, and each consisting of simple products of primes of the form $4p+1$. If $t^2 - 2^\mu . D_1 . D_2^2 . u^2 = -1$ is resoluble, then also is $t^2 - Du^2 = -1$.

We are thus enabled to obtain with ease results of some generality. For example, $2 . 5$ and $2 . 5^2$ are both of the form t^2+1. Hence $2 . 5^m$ is a number D. Again, $2 . 17$, $2 . 17^2$ are not numbers of the kind D. Hence $2 . 17^m$ is not such a number. Similarly, $2 . 13^m$ is a number D, and so is $2 . 29^m$; but $2 . 41^{2m}$ is not such a number, though $2 . 41^{2m+1}$ is. Again, $5^{2m+1} . 13^{2p+1} . 17^{2q+1}$ is a number D, but $5^{2m} . 13^{2p+1} . 17^{2q+1}$ is not; and so on.

Formerly mathematicians not unfrequently gave general expressions for such numbers, but these lists have not been of much use; M. Stern gives a good many cases ("Theorie der Kettenbrüche," 1834, p. 192). Those of his list which relate to numbers of the kind D are of the type $(P+mr)^2 + (Q+ms)^2$, where $Ps - Qr = \pm 1$, $r^2 + s^2 = u^2$, or of the type

$$[2P + (2m+1)\, r]^2 + [2Q + (2m+1)\, s]^2,$$

the last form being the most noteworthy, because the period of the corresponding continued fraction contains a linear function of m. It is plain that if

$$P's - Q'r = \pm 2, \quad r^2 + s^2 = u^2,$$

then $D = P'^2 + Q'^2$ will satisfy $t^2 - Du^2 = -4$, and if P', Q' are respectively odd and even, or even and odd, $t^2 - Du^2 = -1$ will be resoluble; we thus get the above type.

9. We are naturally led to consider these numbers in connection with Pythagorean triangles. Writing r for $2\beta\delta$, s for $\beta^2 - \delta^2$, we have the equations

$$Ps-Qr=\pm\sigma, \quad P^2+Q^2=D, \quad Pr+Qs=t\sigma,$$

$$\left(\frac{r}{\sigma}\right)^2+\left(\frac{s}{\sigma}\right)^2=u^2, \quad t^2-Du^2=-1,$$

by combining which, and using different sets of solutions, a variety of relations are established. The decomposition P^2+Q^2 depends on the factor σ of D. Suppose that β and δ, and therefore r, s, t, u, are derived from a part of the continued fraction terminating in the first period, and that R_q, S_q, T_q, U_q are the corresponding numbers derived from a part of the fraction terminating in a period of rank q.

Then $$(t-\sqrt{D}\,u)^{2q-1}=T_q-\sqrt{D}\,U_q,$$

$$P=\frac{D}{\sigma}\frac{R_{2p}+R_{2q+1}}{T_{2p}+T_{2q+1}}=\frac{D}{\sigma}\frac{R_{2p}-R_{2q}}{T_{2p}-T_{2q}}=\frac{D}{\sigma}\frac{R_{2p+1}-R_{2q+1}}{T_{2p+1}-T_{2q+1}},$$

$$Q=\frac{D}{\sigma}\frac{S_{2p}+S_{2q+1}}{T_{2p}+T_{2q+1}}=\frac{D}{\sigma}\frac{S_{2p}-S_{2q}}{T_{2p}-T_{2q}}=\frac{D}{\sigma}\frac{S_{2p+1}-S_{2q+1}}{T_{2p+1}-T_{2q+1}}.$$

Again, if r, s, σ are given, and (P', Q'), (P'', Q'') are two different solutions of $Ps-Qr=\pm\sigma$, so that $P'r+Q's=t'\sigma$, $P''r+Q''s=t''\sigma$, we have $$P''-P'=(t''-t')\frac{r}{\sigma u^2}, \quad Q''-Q'=(t''-t')\frac{s}{\sigma u^2};$$

or, if we take (P_1, Q_1), a solution of $Ps-Qr=\mp\sigma$,

$$P_1-P'=(t''+t')\frac{r}{\sigma u^2}, \quad Q_1-Q'=(t''+t')\frac{s}{\sigma u^2},$$

and $$\frac{P''-P'}{Q''-Q'}=\frac{P_1+P'}{Q_1+Q'}=\frac{r}{s}.$$

The expression $\frac{P}{r}-\frac{Q}{s}=\pm\frac{\sigma}{rs}$ is a kind of measure of similarity with regard to the triangle whose sides are D, P, Q, and the triangle whose sides are σ, u, r, s. If $\frac{P}{r}-\frac{Q}{s}=0$, the triangles would be similar, and D a perfect square; when therefore r, s are given, the nearest approach to similarity is given by principal solutions, in which case $\sigma=1$.

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* *e.g.* $D=5\,.\,13$, $\sigma=5$, $t=8$, $u=1$, $(8-\sqrt{65})^3=2072-\sqrt{65}\,.\,257$,

$$R_1^2+S_1^2=r^2+s^2=4^2+3^2, \quad R_2^2+S_2^2=(5\,.\,257)^2=1116^2+637^2,$$

$$\frac{65}{5}\frac{3+637}{8+2072}=4, \quad \frac{65}{5}\cdot\frac{4+1116}{8+2072}=7, \quad 4^2+7^2=65.$$

The continued fraction represents $\frac{\sqrt{65}}{5}$.

10. I may remark that a great number of linear relations exist among the component numbers taken in threes. For if a^2+b^2 is a factor of $P^2+Q^2 < \sqrt{P^2+Q^2}$, we may write

$$a+ib \,.\, A+iB = P+iQ,$$

and therefore

$$\frac{(a-ib)^2\,(P+iQ)}{a^2+b^2} = P'+iQ',$$

giving

$$(a^2-b^2)\,P+2abQ-(a^2+b^2)\,P' = 0,$$
$$(a^2-b^2)\,Q-2abP-(a^2+b^2)\,Q' = 0\ ;$$

and thence arise vanishing determinants connecting the component numbers in sets.

If we eliminate successively Q, P, we get

$$(a^2-b^2)\,P'-2abQ'-(a^2+b^2)\,P = 0,$$

or

$$(a^2-b^2)\,Q'+2abP'-(a^2+b^2)\,Q = 0.*$$

With regard to those numbers which are sums of two squares, but are not amenable to the foregoing treatment, it is not easy to find a set of forms depending on the factors which will give all the decompositions. If we know one decomposition corresponding to a form $(\pm a,\ b,\ \mp a)$ of odd determinant,† we arrive at another decomposition $(\pm d,\ c\mp d)$ in the same period ("Proceedings" VII., p. 204). These forms will not belong to ambiguous classes. There must therefore be at least two decompositions, and, as is well known, the number cannot be an odd power of a prime. Since we can arrive at any possible decomposition by some continued fraction (excepted always when the number is a perfect square), it is no doubt unsatisfactory that we cannot determine by a general rule the proper forms to set out with. I am not aware, however, that any such rule is known. The general case seems to depend on the solution of $t^2-Du^2 = -k^2$, where k is an integer.

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* This would have been better put as follows:—

If in integers

$$Ax+By-Cz = 0,$$
$$A^2+B^2-C^2 = 0,$$

A, B, C being relatively prime, then

$$x^2+y^2 = z^2+k^2 = CN.$$

† If the determinant is even, so that a, b are odd, the second form arrived at might be $(\pm b,\ a,\ \mp b)$, giving the same decomposition, and it is for this reason that the double of a prime of the form $4p+1$ is not necessarily a number D (*e.g.*, 2, 17) although there is only one decomposition. Since this paper was read I have found that, if D is the double of an odd power of a prime of the form $8m+5$, $x^2-Dy^2 = -1$ is resoluble.

Generalised Form of Certain Series. By J. W. L. Glaisher, M.A., F.R.S.

[*Read May 9th*, 1878.]

1. The results referred to in the title are consequences of the identical equation

$$\left(1+\frac{p}{p}x+\frac{p\,.\,p+2}{p\,.\,p+1}\frac{x^2}{2!}+\frac{p\,.\,p+2\,.\,p+4}{p\,.\,p+1\,.\,p+2}\frac{x^3}{3!}+\&c.\right)e^{-x}$$
$$=1+\frac{1}{p+1}\frac{x^2}{2}+\frac{1}{p+1\,.\,p+3}\frac{x^4}{2^2.2!}+\frac{1}{p+1\,.\,p+3\,.\,p+5}\frac{x^6}{2^3.3!}+\&c.$$
$$\ldots\ldots(1);$$

and it is convenient to proceed to them at once, leaving the demonstration of (1) to § 4.

It may be observed that, in (1), if p is equal to a negative integer, say if $p=-2i$, then in the series that multiplies e^{-x} there will be i zero terms; for the coefficient of x^{i+1} involves the factor $p+2i$ in the numerator, and therefore vanishes, as also does every term up to x^{2i+1}, in which the factor $p+2i$ first makes its appearance in a denominator, and can therefore be divided out from the numerator and denominator; this factor also can be divided out from all the succeeding terms. Thus the terms in x^{i+1}, x^{i+2}, ... x^{2i} disappear. It follows, therefore, that in the product

$$\left(1+\frac{1}{p+1}\frac{x^2}{2}+\frac{1}{p+1\,.\,p+3}\frac{x^4}{2^2.2!}+\frac{1}{p+1\,.\,p+3\,.\,p+5}\frac{x^6}{2^3.3!}+\&c.\right)e^x,$$

if $p=-2i$, there are no terms in x^{i+1}, x^{i+2}, ... x^{2i}.

If $p=$ a negative uneven integer, both sides of (1) become infinite: this case is supposed to be excluded in all that follows.

2. Since the right-hand side of (1) only involves x^2, it follows that

$$\left(1+x+\frac{p+2}{p+1}\frac{x^2}{2!}+\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^3}{3!}+\&c.\right)e^{-x}$$
$$=\left(1-x+\frac{p+2}{p+1}\frac{x^2}{2!}-\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^3}{3!}+\&c.\right)e^x,$$

where p has any value; whence

$$e^{2x}=\frac{e^x}{e^{-x}}=\frac{1+x+\frac{p+2}{p+1}\frac{x^2}{2!}+\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^3}{3!}+\&c.}{1-x+\frac{p+2}{p+1}\frac{x^2}{2!}-\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^3}{3!}+\&c.}\quad\ldots\ldots\ldots(2).$$

Putting $p=0$ in this expression, the numerator $=\frac{1}{2}(1+e^{2x})$, and the denominator $=\frac{1}{2}(1+e^{-2x})$. Putting $p=\infty$, the numerator and denominator become respectively $=e^x$ and e^{-x}.

From (2) we have

$$\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{x+\frac{p+2}{p+1}\frac{x^3}{3!}+\frac{p+2\ldots p+8}{p+1\ldots p+4}\frac{x^5}{5!}+\&c.}{1+\frac{p+2}{p+1}\frac{x^2}{2!}+\frac{p+2\,.\,p+4\,.\,p+6}{p+1\,.\,p+2\,.\,p+3}\frac{x^4}{4!}+\&c.};$$

therefore

$$\tan x=\frac{x-\frac{p+2}{p+1}\frac{x^3}{3!}+\frac{p+2\ldots p+8}{p+1\ldots p+4}\frac{x^5}{5!}-\&c.}{1-\frac{p+2}{p+1}\frac{x^2}{2!}+\frac{p+2\,.\,p+4\,.\,p+6}{p+1\,.\,p+2\,.\,p+3}\frac{x^4}{4!}-\&c.}\qquad\ldots\ldots(3),$$

which is true for all values of p. The cases of $p=0$ and $p=\infty$ correspond respectively to the formulæ

$$\tan x=\frac{\frac{1}{2}\sin 2x}{\frac{1}{2}(1+\cos 2x)},\qquad \tan x=\frac{\sin x}{\cos x}.$$

For $p=1$, (3) gives

$$\tan x=\frac{x-\frac{3\,.\,5}{2\,.\,3}\frac{x^3}{3!}+\frac{3\,.\,5\,.\,7\,.\,9}{2\,.\,3\,.\,4\,.\,5}\frac{x^5}{5!}-\&c.}{1-\frac{3}{2}\frac{x^2}{2!}+\frac{3\,.\,5\,.\,7}{2\,.\,3\,.\,4}\frac{x^4}{4!}-\&c.};$$

whence, putting $2x$ for x,

$$\tan 2x=\frac{\frac{2!}{(1!)^3}x-\frac{6!}{(3!)^3}x^3+\frac{10!}{(5!)^3}x^5-\&c.}{1-\frac{4!}{(2!)^3}x^2+\frac{8!}{(4!)^3}x^4-\&c.}.$$

3. From (2) it follows that

$$\frac{2x}{e^{2x}-1}=\frac{1-x+\frac{p+2}{p+1}\frac{x^2}{2!}-\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^3}{3!}+\&c.}{1+\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^2}{3!}+\frac{p+2\ldots p+8}{p+1\ldots p+4}\frac{x^4}{5!}+\&c.}$$

$$=1-x+\frac{B_1}{2!}2^2x^2-\frac{B_2}{4!}2^4x^4+\&c.,$$

$B_1, B_2, B_3\ldots$ being the Bernoullian numbers; whence

$$\frac{1+\frac{p+2}{p+1}\frac{x^2}{2!}+\frac{p+2\,.\,p+4\,.\,p+6}{p+1\,.\,p+2\,.\,p+3}\frac{x^4}{4!}+\&c.}{1+\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^2}{3!}+\frac{p+2\ldots p+8}{p+1\ldots p+4}\frac{x^4}{5!}+\&c.}$$

$$= 1 + \frac{B_1}{2!} 2^2 x^2 - \frac{B_2}{4!} 2^4 x^4 + \frac{B_3}{6!} 2^6 x^6 - \&c.$$

Multiplying up by the denominator of the left-hand expression, and equating the coefficients of x^{2n}, we have

$$2^{2n} B_n - \frac{p+2 \,.\, p+4}{p+1 \,.\, p+2} \frac{2n\,(2n-1)}{3!} 2^{2n-2} B_{n-1}$$

$$+ \frac{p+2 \,\ldots\, p+8}{p+1 \,\ldots\, p+4} \frac{2n \,\ldots\, 2n-3}{5!} 2^{2n-4} B_{n-2} \,\ldots\ldots$$

$$\ldots\ldots + (-)^{n+1} \frac{p+2 \,.\, p+4 \,\ldots\, p+4n}{p+1 \,.\, p+2 \,\ldots\, p+2n} \frac{1}{2n+1}$$

$$= (-)^{n+1} \frac{p+2 \,.\, p+4 \,\ldots\, p+4n-2}{p+1 \,.\, p+2 \,\ldots\, p+2n-1},$$

which is true independently of the value of p.

As an example, put $n = 2$, and this formula gives

$$2^4 B_2 - \frac{p+2 \,.\, p+4}{p+1 \,.\, p+2} 2^3 B_1 - \frac{p+2 \,\ldots\, p+8}{p+1 \,\ldots\, p+4} \,.\, \frac{1}{5} = - \frac{p+2 \,.\, p+4 \,.\, p+6}{p+1 \,.\, p+2 \,.\, p+3},$$

viz., $$2^4 B_2 \,.\, p+1 \,.\, p+3 - 2^3 B_1 \,.\, p+4 \,.\, p+3 - (p+6 \,.\, p+8) \tfrac{1}{5}$$
$$= - (p+4 \,.\, p+6),$$

viz., $2^4 B_2 (p^2+4p+3) - 2^3 B_1 (p^2+7p+12) + \frac{1}{5} (4p^2+36p+72) = 0$;

whence, equating the coefficients of the powers of p to zero,

$$\begin{aligned} 2^4 B_2 \quad - 2^3 B_1 \quad + \tfrac{4}{5} &= 0, \\ 2^4 B_2 \,.\, 4 - 2^3 B_1 \,.\, 7 + \tfrac{36}{5} &= 0, \\ 2^4 B_2 \,.\, 3 - 2^3 B_1 \,.\, 12 + \tfrac{72}{5} &= 0; \end{aligned}$$

which, since $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, are at once seen to be true.

4. In order to prove (1), we observe that the coefficient of x^n on the left-hand side

$$= (-)^n \left(\frac{1}{n!} - \frac{p}{p} \frac{1}{(n-1)!} + \frac{p \,.\, p+2}{p \,.\, p+1} \frac{1}{(n-2)!\,2!} \,\ldots\ldots \right.$$

$$\left. \ldots\ldots + (-)^n \frac{p \,.\, p+2 \,\ldots\, p+2n-2}{p \,.\, p+1 \,\ldots\, p+n-1} \frac{1}{n!} \right)$$

$$= (-)^n \left(\frac{1}{n!} - \frac{\frac{1}{2}p}{p} \frac{2}{(n-1)!} + \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1}{p \,.\, p+1} \frac{2^2}{(n-2)!\,2!} \,\ldots\ldots \right.$$

$$\left. \ldots\ldots + (-)^n \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1 \,\ldots\, \frac{1}{2}p+n-1}{p \,.\, p+1 \,\ldots\, p+n-1} \frac{2^n}{n!} \right)$$

$$= (-)^n \frac{1}{p \,.\, p+1 \,\ldots\, p+n-1}$$
$$\times \Big(\frac{p \,.\, p+1 \,\ldots\, p+n-1}{n!} - \tfrac{1}{2}p \,.\, 2 \,.\, \frac{p+1 \,\ldots\, p+n-1}{(n-1)!}$$
$$+ \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1}{2!} \,.\, 2^2 \,.\, \frac{p+2 \,\ldots\, p+n-1}{(n-2)!} \,\ldots\ldots$$
$$\ldots\ldots + (-)^n \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1 \,\ldots\, \frac{1}{2}p+n-1}{n!} \, 2^n \Big);$$

and $(-)^n \times$ the expression in brackets = the coefficient of t^n in

$$(1+t)^{-p} + \tfrac{1}{2}p \,.\, 2t\,(1+t)^{-p-1} + \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1}{2!} \, 2^2 t^2 \,(1+t)^{-p-2} + \&c.,$$

viz., in $$(1+t)^{-p}\Big(1 - \frac{2t}{1+t}\Big)^{-\frac{1}{2}p}, \; = (1-t^2)^{-\frac{1}{2}p}.$$

Thus, if n be uneven, the coefficient of x^n is zero; and if n be even, the coefficient of x^n

$$= \frac{1}{p \,.\, p+1 \,\ldots\, p+n-1} \,.\, \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1 \,\ldots\, \frac{1}{2}p+\frac{1}{2}n-1}{(\frac{1}{2}n)!}$$
$$= \frac{1}{p+1 \,.\, p+3 \,\ldots\, p+n-1} \frac{1}{2^{\frac{1}{2}n} (\frac{1}{2}n)!}.$$

This proof does not apply as it stands when $p = -2i$, but it can be extended so as to include this case by taking $p = -2i+h$, and making h indefinitely small. The equation (1) can, however, be proved for all values of p without the use of limits, as follows:—

Consider the coefficient of x^n in the product

$$\Big(1 + \frac{1}{p+1} \frac{x^2}{2} + \frac{1}{p+1 \,.\, p+3} \frac{x^4}{2^2 \,.\, 2!} + \frac{1}{p+1 \,.\, p+3 \,.\, p+5} \frac{x^6}{2^3 \,.\, 3!} + \&c.\Big) e^x.$$

First, suppose n even $= 2m$; the coefficient of x^{2m}

$$= \frac{1}{(2m)!} + \frac{1}{p+1} \frac{1}{2} \frac{1}{(2m-2)!} + \frac{1}{p+1 \,.\, p+3} \frac{1}{2^2 \,.\, 2!} \frac{1}{(2m-4)!} \cdots$$
$$\ldots + \frac{1}{p+1 \,.\, p+3 \,\ldots\, p+2m-1} \frac{1}{2^m \,.\, m!}$$
$$= \frac{1}{2m \,.\, 2m-1 \,\ldots\, m+1} \; \frac{1}{p+1 \,.\, p+3 \,\ldots\, p+2m-1}$$
$$\times \Big(\frac{p+1 \,.\, p+3 \,\ldots\, p+2m-1}{m!} + \frac{p+3 \,\ldots\, p+2m-1}{(m-1)!} \frac{2m \,.\, 2m-1}{m} \frac{1}{2}$$
$$+ \frac{p+5 \,\ldots\, p+2m-1}{(m-2)!} \; \frac{2m \,\ldots\, 2m-3}{m \,.\, m-1} \frac{1}{2^2} \frac{1}{2!} \cdots$$
$$\ldots + \frac{2m \,.\, 2m-1 \,\ldots\, 1}{m!} \frac{1}{2^m \,.\, m!} \Big),$$

and the expression in brackets

$$= 2^m \left(\frac{\frac{1}{2}(p+1) \cdot \frac{1}{2}(p+1)+1 \ \ldots \ \frac{1}{2}(p+1)+m-1}{m\,!} \right.$$
$$+ \frac{\frac{1}{2}(p+3) \ \ldots \ \frac{1}{2}(p+3)+m-2}{(m-1)\,!} (m-\tfrac{1}{2})$$
$$\left. + \frac{\frac{1}{2}(p+5) \ \ldots \ \frac{1}{2}(p+5)+m-3}{(m-2)\,!} \frac{m-\frac{1}{2} \cdot m-\frac{3}{2}}{2\,!} \ldots + \frac{m-\frac{1}{2} \cdot m-\frac{3}{2} \ldots \frac{1}{2}}{m\,!} \right)$$

$= 2^m$. coefficient of t^m in

$$(1-t)^{-\frac{1}{2}(p+1)} + (m-\tfrac{1}{2})\, t\, (1-t)^{-\frac{1}{2}(p+3)} + \frac{m-\frac{1}{2} \cdot m-\frac{3}{2}}{2\,!}\, t^2\, (1-t)^{-\frac{1}{2}(p+5)} + \&c.,$$

viz., in $(1-t)^{-\frac{1}{2}(p+1)} \left(1+\frac{t}{1-t}\right)^{m-\frac{1}{2}}$, $= (1-t)^{-\frac{1}{2}(p+1)-m+\frac{1}{2}}$, $= (1-t)^{-\frac{1}{2}p-m}$.

Thus, 2^m . coefficient of t^m

$$= 2^m \frac{\frac{1}{2}p+m \cdot \frac{1}{2}p+m+1 \ \ldots \ \frac{1}{2}p+2m-1}{m\,!}$$
$$= \frac{p+2m \cdot p+2m+2 \ \ldots \ p+4m-2}{m\,!},$$

and therefore the coefficient of x^{2m}

$$= \frac{p+2m \cdot p+2m+2 \ \ldots \ p+4m-2}{p+1 \cdot p+3 \ \ldots \ p+2m-1} \frac{1}{(2m)\,!}$$
$$= \frac{p \cdot p+2 \cdot p+4 \ \ldots \ p+4m-2}{p \cdot p+1 \cdot p+2 \ \ldots \ p+2m-1} \frac{1}{(2m)\,!}.$$

Secondly, suppose n uneven $= 2m+1$, it can be shown, as above, that the coefficient of x^{2m+1}

$$= \frac{1}{2m+1 \cdot 2m \ \ldots \ m+1} \cdot \frac{1}{p+1 \cdot p+3 \ \ldots \ p+2m-1} \cdot 2^m \cdot T,$$

where $T =$ the coefficient of t^m in

$$(1-t)^{-\frac{1}{2}(p+1)} + (m+\tfrac{1}{2})\, t\, (1-t)^{-\frac{1}{2}(p+3)} + \frac{m+\frac{1}{2} \cdot m-\frac{1}{2}}{2\,!}\, t^2\, (1-t)^{-\frac{1}{2}(p+5)} + \&c.,$$

$$= (1-t)^{-\frac{1}{2}(p+1)} \left(1+\frac{t}{1-t}\right)^{m+\frac{1}{2}} = (1-t)^{-\frac{1}{2}p-m-1}.$$

Thus
$$T = \frac{\frac{1}{2}p+m+1 \cdot \frac{1}{2}p+m+2 \ \ldots \ \frac{1}{2}p+2m}{p+1 \cdot p+3 \ \ldots \ p+2m-1},$$

whence the coefficient of x^{2m+1}

$$= \frac{p \cdot p+2 \ \ldots \ p+4m}{p \cdot p+1 \ \ldots \ p+2m} \frac{1}{(2m+1)\,!}.$$

5. I may mention that the right-hand side of (1) is a Bessel's function; for

$$J_n(z) = \frac{z^n}{2^n\Gamma(n+1)}\left(1 - \frac{z^2}{2\,.\,2n+2} + \frac{z^4}{2\,.\,4\,.\,2n+2\,.\,2n+4} - \&c.\right),$$

and we obtain the series in (1) on putting $n = \frac{1}{2}(p-1)$ and $z = xi$. It was, in fact, in connexion with the different particular integrals of the differential equation

$$\frac{d^2u}{dx^2} - a^2u = \frac{n\,.\,n+1}{x^2}\,u,$$

which is a transformation of the equation of Bessel's functions, that I met with the equation (1).

A Bessel's function is, as is known, a limiting form of a hypergeometric series, and the equation (1) may, in fact, be written

$$F\left(a,\ \tfrac{1}{2}p,\ p,\ \frac{2x}{a}\right)e^{-x} = F\left(a,\ \beta,\ \tfrac{1}{2}p+\tfrac{1}{2},\ \frac{x^2}{4a\beta}\right),$$

where a, a, β are each of them infinite; or, if we please,

$$F\left(a,\ \tfrac{1}{2}p,\ p,\ \frac{2x}{a}\right)e^{-x} = F\left(a,\ a,\ \tfrac{1}{2}p+\tfrac{1}{2},\ \frac{x^2}{4a^2}\right),$$

where a is infinite.

On a new method of determining the Differential Resolvents of Algebraical Equations. *By* Robert Rawson, Associate of I.N.A., Hon. Member of the Literary and Philosophical Society, Manchester.

[*Read June 13th*, 1878.]

The theory of *differential resolvents* of algebraical equations appears to have originated in a short sketch of a theory of transcendental roots by Sir James Cockle, F.R.S., published in the "Philosophical Magazine" for August, 1860. Eighteen months later, the subject was considered in a series of suggestive papers on the theory of transcendental solutions, by the Rev. Robert Harley, F.R.S., published in the "Proceedings" of the Literary and Philosophical Society of Manchester, Vol. ii., pp. 181—184, 199—203, 237—241.

The leading idea is, to determine, from a given algebraical equation, a linear differential equation which is satisfied by the roots of the given

algebraical equation from which it has been derived. The subject has excited considerable interest among some of our most eminent English mathematicians, more particularly Professor Cayley, Professor Boole, and Messrs. Spottiswoode, Russell, and Todhunter.

In the memoirs of the Literary and Philosophical Society of Manchester there are several valuable papers on differential resolvents.

1. "Note on a Differential Equation." By A. Cayley, Esq., M.A., F.R.S., &c. In this paper Lagrange's theorem is applied, for the first time, to determine the differential resolvent of $y^n - ny + (n-1)x = 0$.

2. "Note on Differential Resolvents." By William Spottiswoode, M.A., F.R.S., &c. In this paper the resolvents of a general quadratic and cubic are determined by the application of determinants.

The reader of this paper should observe two corrections, viz.,

The 10th line from top of p. 231, for a'^2a^2b read $(d')^2a^2b$,

The 11th do. do. for $3a'b'ad^2$ read $a'b'(3ad^2 - 9bcd)$.

3. "On a certain class of Linear Differential Equations." By the Rev. Robert Harley, F.R.S. In this paper Mr. Harley has discussed two classes of equations, from the quadratic to the quintic, both inclusive.

4. "On the solution of the Differential Resolvent." By W. H. L. Russell, A.B. The solution is by means of a definite integral.

In the Transactions of the Royal Society, May 1864, there is a paper on this subject by Boole, entitled "On the Differential Equations which determine the form of the roots of Algebraical Equations." In this paper the method of Professor Cayley is employed, and the resolvent is solved by a definite integral. It is remarkable that neither Boole nor Russell has noticed the theorem of M. Perseval (*Mem. Inst*, Tome I.) as given by Moseley in his article on Definite Integrals, "Encyclopædia Metropolitana," page 530. This theorem expresses the roots of $y = a + \phi(y)$ by a definite integral.

Mr. Todhunter has devoted an article to the subject of Differential Resolvents in his supplementary volume of Boole's "Differential Equations," pp. 190—199. The materials of the article were supplied by Mr. Harley.

Although the germs of the theory were first given by Sir James Cockle, yet, as that eminent analyst remarks, in a paper on Differential Equations, published by the Royal Society of Victoria, in 1866, Mr. Harley, "by his earlier researches into the forms of the differential resolvents of certain trinomial algebraical equations, obtained results which not only excited attention at the time, but which have also, to a great extent, determined the current of subsequent research. Traces of Mr. Harley's investigations appear in almost every paper that has since been published."

General Views.

1. Let ϕ be a function of two independent variables x, y. Then, assuming that $\log\left(\frac{\phi}{y}\right)$ can be expanded in descending powers of y, we may write

$$\log\left(\frac{\phi}{y}\right) = \Sigma\frac{u_n}{y^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where u_n is a function of x, such that

$$u_n = -\frac{\beta_1^n}{n}, \text{ where } \beta_1 \text{ is a root of } \phi = 0 \ldots\ldots\ldots\ldots (2).$$

See Murphy's theorem, as given by Todhunter in his theory of Equations, and also by Peacock in his "Report of Certain Branches of Analysis," British Association, Vol 3.

If ϕ represents an algebraical function of x, y, and of (m) dimensions in y, having $\beta_1, \beta_2, \ldots \beta_m$ for its roots, then, by equations (1) and (2),

we have $$\log\frac{\phi}{y\,(y-\beta_1)(y-\beta_2)\ \ldots\ (y-\beta_{m-1})} = \Sigma\frac{u_n}{y^n} \ldots\ldots\ldots\ldots (3),$$

where $$u_n = -\frac{\beta_m^n}{n},\ \beta_m \text{ being the } m^{\text{th}} \text{ root of } \phi = 0 \ldots\ldots\ldots\ldots (4).$$

Now Mr. Harley has shewn that any root of $\phi = 0$ will satisfy a certain linear differential equation of the $(m-1)^{\text{th}}$ order, derived from $\phi = 0$, which differential equation he designates the *resolvent* of $\phi = 0$. The object then, in what follows is to find, from a given equation $\phi = 0$, its differential resolvent by means of Murphy's theorem. The integration of this resolvent gives the roots of the equation from which it has been derived.

2. If $\phi = y^m + ay^r + bx$, a case considered by Mr. Harley, then

$$\log\frac{y^m + ay^r + bx}{y^s} = \Sigma\frac{u_n}{y^n} \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where $$u_n = -\frac{1}{n}\{\beta_1^n + \beta_2^n + \ldots \beta_s^n\} \ldots\ldots\ldots\ldots\ldots (6),$$

$\beta_1, \beta_2, \ldots \beta_s$ are s of the m roots of $y^m + ay^r + bx = 0$.

Differentiate (5) as follows, then

$$\frac{b}{y^m + ay^r + bx} = \Sigma\frac{u_n'}{y^n} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\frac{my^m + ary^r}{y^m + ay^r + bx} = s - \Sigma\frac{nu_n}{y^n} \ldots\ldots\ldots\ldots\ldots (8),$$

$$\frac{b(my^m+ary^r)}{y^m+ay^r+bx} = \Sigma \frac{nu'_n}{y^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where differentiation is denoted by an accent.

From (7) and (8) it follows that

$$(my^m+ary^r)\left\{\Sigma \frac{u'_n}{y^n}\right\} = b\left\{s-\Sigma \frac{nu_n}{y^n}\right\}.$$

Equate the coefficients of $\frac{1}{y^n}$, then

$$mu'_{n+m}+aru'_{n+r}+bnu_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

From (7) and (9) we obtain

$$(my^m+ary^r)\left\{\Sigma \frac{u'_n}{y^n}\right\} = (y^m+ay^r+bx)\left\{\Sigma \frac{nu'_n}{y^n}\right\}.$$

Equate the coefficients of $\frac{1}{y^n}$, then

$$mu'_{n+m}+aru'_{n+r} = (n+m)\,u'_{n+m}+a(n+r)\,u'_{n+r}+nbxu'_n,$$

therefore
$$u'_{n+m}+au'_{n+r}+bxu'_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

From (10) and (11) it readily follows that

$$u'_{n+r} = -\frac{bm}{a(m-r)}\left\{xu'_n-\frac{n}{m}u_n\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

$$u'_{n+m} = \frac{br}{m-r}\left\{xu'_n-\frac{n}{r}u_n\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The following notation will be used for convenience, viz.,

$$u'_n = \frac{du_n}{dx}, \quad u''_n = \frac{d^2u_n}{dx^2} \ldots u^m_n = \frac{d^mu_n}{ax^m}.$$

Now
$$u^q_{n+qr} = u^q_{n+pm} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

when
$$pm = qr \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

It will be observed that the equations (12) and (13) are independent of s; and therefore that any linear differential equation in u_n will be satisfied by the n^{th} power of any root of $\phi = 0$. Hence, by making $n = 1$, such differential equation becomes the differential resolvent of the algebraical equation.

In equation (14), put $p = r$ and $q = m$, then (15) is satisfied; or put p for the numerator and q for the denominator of the fraction $\frac{r}{m}$ when reduced to its lowest terms, then also (15) is satisfied. Therefore a linear differential equation of the m^{th} order can be found from (12), (13), (14), which is satisfied by the n^{th} power of any root of

$$y^m+ay^r+bx = 0.$$

3. The following is the application of (12), (13) to a cubic, viz., $y^3+ay^2+bx=0$. In this case, $m=3$, $r=2$, $p=2$, $q=3$, and (12) and (13) become

$$u'_{n+2}=-\frac{3b}{a}\left\{x u'_n-\frac{n}{3}u_n\right\} \quad \text{.....(16),}$$

$$u'_{n+3}=\quad 2b\left\{x u'_n-\frac{n}{2}u_n\right\} \quad \text{.....(17).}$$

From (16) we have, by differentiation,

$$u^3_{n+2}=-\frac{3b}{a}\left\{x u^3_n+\left(2-\frac{n}{3}\right)u^2_n\right\};$$

$$\therefore\ u^3_{n+4}=\frac{3^2b^2}{a^2}\left\{x\left[x u^3_n+\left(2-\frac{n}{3}\right)u^2_n\right]+\frac{4-n}{3}\left[x u^2_n+\left(1-\frac{n}{3}\right)u^1_n\right]\right\}$$

(by taking $n+2$ for n)

$$=\frac{3^2b^2}{a^2}\left\{x u^3_n+\frac{10-2n}{3}\cdot x u^2_n+\frac{4-n}{3}\cdot\frac{3-n}{3}u^1_n\right\};$$

$$\therefore\ u^3_{n+6}=-\frac{3^3b^3}{a^3}\left\{x^2\left[x u^3_n+\left(2-\frac{n}{3}\right)u^3_n\right]\right.$$

$$\left.+\frac{6-2n}{3}x\left[x u^2_n+\left(1-\frac{n}{3}\right)u^1_n\right]+\frac{2-n}{3}\cdot\frac{1-n}{3}\left[x u^1_n-\frac{n}{3}u_n\right]\right\}$$

$$=-\frac{3^3b^3}{a^3}\left\{x^3u^3_n+(4-n)x^2u^2_n+\frac{(2-n)(1-n)+2(3-n)^2}{9}\cdot x u^1_n\right.$$

$$\left.-\frac{n}{3}\cdot\frac{1-n}{3}\cdot\frac{2-n}{3}u_n\right\} \quad \text{.....(18).}$$

In like manner, from (17), we have

$$u^3_{n+3}=2b\left\{x u^3_n+\left(2-\frac{n}{2}\right)u^2_n\right\};$$

$$\therefore\ u^3_{n+6}=4b^2\left\{x\left[x u^3_n+\left(2-\frac{n}{2}\right)u^2_n\right]+\frac{1-n}{2}\left[x u^2_n+\left(1-\frac{n}{2}\right)u^1_n\right]\right\}$$

$$=4b^2\left\{x^2u^3_n+\frac{5-2n}{2}\cdot x u^2_n+\frac{1-n}{2}\cdot\frac{2-n}{2}\cdot u^1_n\right\} \quad \text{.....(19).}$$

Equate (18) and (19), then

$$x^2\left(4+\frac{27bx}{a^3}\right)u^3_n+x\left((10-4n)+\frac{27b(4-n)x}{a^3}\right)u^2_n$$

$$+\left\{(1-n)(2-n)+\frac{3b}{a^3}\left((2-n)(1-n)+2(3-n)^2\right)x\right\}u^1_n$$

$$-\frac{bn}{a^3}(1-n)(2-n)u_n=0 \ldots (20).$$

This equation is satisfied by $c\beta_1^n$, where β_1 is any one of the roots of $y^3+ay^2+bx=0$. It can be depressed to an equation of the 2nd order in the cases $n=1, 2, -1, -2, -3$.

Let us consider these cases separately.

When $n=1$, (20) becomes

$$x\left(4+\frac{27bx}{a^3}\right)u_1^3+\left(6+\frac{81bx}{a^3}\right)u_1^2+\frac{24b}{a^3}u_1^1=0 \ldots\ldots\ldots (21).$$

This equation is satisfied by β', or, what is the same thing, by $\frac{d\beta}{dx}$.

Integrate (21), then

$$x\left(4+\frac{27bx}{a^3}\right)u_1^2+\left(2+\frac{27bx}{a^3}\right)u_1^1-\frac{3b}{a^3}u_1=C=\frac{b}{a^2} \ldots\ldots (22),$$

the constant C being determined by putting $x=0$, when u_1^2 and u_1^1 both vanish, and $u_1=-\frac{1}{3}a$.

Equation (22) is the resolvent of $y^3+ay^2+bx=0$, and agrees with the form given by Mr. Harley in the "Manchester Memoirs," Vol. iii., New Series, page 239.

Let $n=2$, then (20) becomes

$$x\left(4+\frac{27bx}{a^3}\right)u_2^3+\left(2+\frac{54bx}{a^3}\right)u_2^2+\frac{6b}{a^3}u_2^1=0 \ldots\ldots\ldots (23);$$

and, by integration,

$$x\left(4+\frac{27bx}{a^3}\right)u_2^2-2u_2^1+\frac{6b}{a^3}u_2=C=-\frac{2b}{a} \ldots\ldots\ldots (24),$$

the constant being determined as in the former case.

Let $n=-1$, then (20) becomes

$$x^2\left(4+\frac{27bx}{a^3}\right)u_{-1}^3+x\left(14+\frac{135bx}{a^3}\right)u_{-1}^2+\left(6+\frac{114bx}{a^3}\right)u_{-1}^1+\frac{6b}{a^3}u_{-1}=0$$
$$\ldots\ldots\ldots (25).$$

This equation is satisfied by the reciprocal of any root of the given cubic $y^3+ay^2+bx=0$.

Integrate (25), and determine the constant as before, then

$$x^2\left(4+\frac{27bx}{a^3}\right)u_{-1}^2+x\left(6+\frac{54bx}{a^3}\right)u_{-1}^1+\frac{6bx}{a^3}u_{-1}=C=0 \ldots (26),$$

an equation which is satisfied by $\frac{1}{\beta_1}$.

Let $n=-2$, then (20) becomes

$$x^2\left(4+\frac{27bx}{a^3}\right)u_{-2}^3+x\left(18+\frac{162bx}{a^3}\right)u_{-2}^2+\left(12+\frac{186bx}{a^3}\right)u_{-2}^1+\frac{24b}{a^3}u_{-2}=0$$
$$\ldots\ldots\ldots (27).$$

Integrate, then,

$$x^2\left(4+\frac{27bx}{a^3}\right)u_{-2}^2+x\left(10+\frac{81bx}{a^3}\right)u_{-2}^1+\left(2+\frac{24bx}{a^3}\right)u_{-2}=C=-\frac{2}{a^2}$$
$$\ldots\ldots\ldots (28),$$

an equation which is satisfied by $\frac{1}{\beta_1^2}$.

Let $n=-3$, then this equation becomes

$$x^2\left(4+\frac{27bx}{a^3}\right)u^3_{-3}+x\left(22+\frac{189bx}{a^3}\right)u^2_{-3}+\left(20+\frac{276bx}{a^3}\right)u^1_{-3}+\frac{60b}{a^3}u_{-3}=0 \quad \ldots\ldots\ldots(29).$$

Integrate, then,

$$x^2\left(4+\frac{27bx}{a^3}\right)u^2_{-3}+x\left(14+\frac{168bx}{a^3}\right)u^1_{-3}+\left(6+\frac{60bx}{a^3}\right)u_{-3}=C=-\frac{6}{a^3} \quad \ldots\ldots\ldots(30).$$

5. *Application of* (12) *and* (13) *to a quartic* $y^4+ay^3+bx=0$.

In this case, $m=q=4$, $r=p=3$.

Differentiate (12) with respect to x, then

$$u^4_{n+3}=-\frac{4b}{a}\left\{xu^4_n+\left(3-\frac{n}{4}\right)u^3_n\right\}$$

$$\therefore\ u^4_{n+6}=\frac{4^2b^2}{a^2}\left\{x\left[xu^4_n+\left(3-\frac{n}{4}\right)u^3_n\right]+\frac{9-n}{4}\left[xu^3_n+\left(2-\frac{n}{4}\right)u^2_n\right]\right\}$$

taking $n+3$ for n,

$$=\frac{4^2b^2}{a^2}\left\{x^2u^4_n+\frac{21-2n}{4}xu^3_n+\frac{9-n}{4}\cdot\frac{8-n}{4}\cdot u^2_n\right\},$$

$$u^4_{n+9}=-\frac{4^3b^3}{a^3}\left\{x^2\left[xu^4_n+\left(3-\frac{n}{4}\right)u^3_n\right]+\frac{15-2n}{4}\cdot x\left[xu^3_n+\left(2-\frac{n}{4}\right)u^2_n\right]+\frac{6-n}{4}\cdot\frac{5-n}{4}\left[xu^2_n+\left(1-\frac{n}{4}\right)u^1_n\right]\right\}$$

$$=-\frac{4^3b^3}{a^3}\left\{x^3u^4_n+\frac{27-3n}{4}x^2u^3_n+\left(\frac{15-2n}{4}\cdot\frac{8-n}{4}+\frac{6-n}{4}\cdot\frac{5-n}{4}\right)xu^2_n+\frac{6-n}{4}\cdot\frac{5-n}{4}\cdot\frac{4-n}{4}\cdot u^1_n\right\},$$

$$u^4_{n+12}=\frac{4^4b^4}{a^4}\left\{x^3\left[xu^4_n+\left(3-\frac{n}{4}\right)u^3_n\right]+\frac{18-3n}{4}\cdot x^2\left[xu^3_n+\left(2-\frac{n}{4}\right)u^2_n\right]+\left(\frac{9-2n}{4}\cdot\frac{5-n}{4}+\frac{3-n}{4}\cdot\frac{2-n}{4}\right)x\left[xu^2_n+\left(1-\frac{n}{4}\right)u^1_n\right]+\frac{3-n}{4}\cdot\frac{2-n}{4}\cdot\frac{1-n}{4}\left[xu^1_n-\frac{n}{4}u_n\right]\right\}$$

$$=\frac{4^4b^4}{a^4}\left\{x^4u^4_n+\frac{30-4n}{4}x^3u^3_n\right.$$

$$+\left(\frac{18-n}{4}\cdot\frac{8-n}{4}+\frac{9-2n}{4}\cdot\frac{5-n}{4}+\frac{3-n}{4}\cdot\frac{2-n}{4}\right)x^2u^2_n$$

$$+\left(\frac{9-2n}{4}\cdot\frac{5-n}{4}\cdot\frac{4-n}{4}+\frac{4-n}{4}\cdot\frac{3-n}{4}\cdot\frac{2-n}{4}+\frac{3-n}{4}\cdot\frac{2-n}{4}\cdot\frac{1-n}{4}\right)xu^1_n$$

$$\left.-\frac{n}{4}\cdot\frac{3-n}{4}\cdot\frac{2-n}{4}\cdot\frac{1-n}{4}u_n\right\} \quad \ldots\ldots\ldots(31).$$

Differentiate (13), then

$$u^4_{n+4} = 3b\left\{xu^4_n + \left(3-\frac{n}{3}\right)u^3_n\right\};$$

$$\therefore\ u^4_{n+8} = 3^2b^2\left\{x\left[xu^4_n + \left(3-\frac{n}{3}\right)u^3_n\right] + \frac{5-n}{3}\left[xu^3_n + \left(2-\frac{n}{3}\right)u^2_n\right]\right\}$$

$$= 3^2b^2\left\{x^2u^4_n + \frac{14-2n}{3}xu^3_n + \frac{5-n}{3}\cdot\frac{6-n}{3}\cdot u^2_n\right\},$$

$$u^4_{n+12} = 3^3b^3\left\{x^2\left[xu^4_n + \left(3-\frac{n}{3}\right)u^3_n\right] + \frac{6-2n}{3}x\left[xu^3_n + \left(2-\frac{n}{3}\right)u^2_n\right]\right.$$
$$\left. + \frac{1-n}{3}\cdot\frac{2-n}{3}\left[xu^2_n + \left(1-\frac{n}{3}\right)u^1_n\right]\right\}$$

$$= 3^3b^3\left\{x^3u^4_n + (5-n)x^2u^3_n + \left(\frac{6-2n}{3}\cdot\frac{6-n}{3} + \frac{1-n}{3}\cdot\frac{2-n}{3}\right)xu^2_n\right.$$
$$\left. + \frac{1-n}{3}\cdot\frac{2-n}{3}\cdot\frac{3-n}{3}\cdot u^1_n\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(32).$$

Hence (14) becomes

$$x^3\left(3^3 - \frac{4^4bx}{a^4}\right)u^4_n + x^2\left(3^3(5-n) - \frac{4^3b\,(30-4n)\,x}{a^4}\right)u^3_n$$
$$+ \left\{6(3-n)(6-n) + 3(1-n)(2-n) - \frac{b}{a^4}\left((18-3n)(8-n)\right.\right.$$
$$\left.\left. + (9-2n)(5-n) + (3-n)(2-n)\right)4^2x\right\}xu^2_n$$
$$+ \left\{(1-n)(2-n)(3-n) - \frac{b}{a^4}\left((9-2n)(5-n)(4-n)\right.\right.$$
$$\left.\left. + (4-n)(3-n)(2-n) + (3-n)(2-n)(1-n)\right)4x\right\}u^1_n$$
$$+ \frac{b}{a^4}n(3-n)(2-n)(1-n)\,u_n = 0 \quad \ldots\ldots(33).$$

When $(n-1)(n-2)(n-3) = 0$, that is to say, when $n = 1, 2$, or 3, this equation obviously admits of being depressed to an equation of the third order. It can be shown also to admit of depression in the cases $n = -1, -2, -3, -4$. For let

$$x^3\left(3^3 - \frac{4^4bx}{a^4}\right)u^3_n + (A_1x^2 + A_2x^3)\,u^2_n + (A_3x + A_4x^2)\,u^1_n$$
$$+ (A_5 + A_6x)\,u_n + C = 0 \quad \ldots\ldots(34),$$

where A_1, A_2, A_3, &c. are constants to be determined, be the first integral of (33). Differentiate and compare with (33), then

$$x^3\left(3^3 - \frac{4^4bx}{a^4}\right)u^4_n + \left.\begin{matrix} 3^4x^2 - \dfrac{4^5bx^3}{a^4} \\ A_1x^2 + A_2x^3 \end{matrix}\right\}u^3_n + \left.\begin{matrix} 2A_1x + 3A_2x^2 \\ A_3x + A_4x^2 \end{matrix}\right\}u^2_n$$
$$+ \left.\begin{matrix} A_3 + 2A_4x^2 \\ A_5 + A_6x^2 \end{matrix}\right\}u^1_n + A_6u_n = 0;$$

$$\therefore \quad A_1 = 3^2(2-n), \qquad A_2 = \frac{4^3 b}{a^4}(4n-14),$$

$$A_3 = 3(1-3n+3n^2), \qquad A_4 = \frac{4^2 b}{a^4}(-27+18n-6n^2),$$

$$A_5 = -n+15n^2-10n^3, \qquad A_6 = \frac{4b}{a^4}(6+14n+6n^2+4n^3).$$

But $A_6 = \frac{b}{a^4} n(3-n)(2-n)(1-n) = \frac{b}{a^4}(6n-11n^2+6n^3-n^4)$,

and therefore $n^4+10n^3+35n^2+50n+24 = 0$ .....(35),

the roots of which are $n = -1, -2, -3, -4$.

The case $n = 1$ leads to the differential resolvent of the quartic equation $y^4+ay^3+bx = 0$ .....(36).

Here

$$x^2\left(3^3-\frac{4^4bx}{a^4}\right)u_1^4 + x\left(108-\frac{26\,.\,4^3bx}{a^4}\right)u_1^3$$

$$+\left(60-\frac{135\,.\,4^2bx}{a^4}\right)u_1^2 - \frac{360bu_1^1}{a^4} = 0 \text{.....(37).}$$

Integrate, then

$$x^2\left(27-\frac{4^4bx}{a^4}\right)u_1^3 + x\left(54-\frac{14\,.\,4^3bx}{a^4}\right)u_1^2$$

$$+\left(6-\frac{23\,.\,4^2bx}{a^4}\right)u_1^1 + \frac{8b}{a^4}u_1 = C = -\frac{2b}{a^4} \text{ ... (38).}$$

This agrees with the form of the differential resolvent of (36), given by Mr. Harley in his memoir in the "Transactions" of the Manchester Society.

6. By the foregoing method we may determine generally a linear differential equation, which is satisfied by the n^{th} power of any root of the algebraical equation

$$y^m+ay^r+bx = 0.$$

This differential equation will be of the order q or p, according as m or r is the greater exponent, q being the numerator and p the denominator of the fraction $\frac{m}{r}$ reduced to its lowest terms.

By differentiation we obtain, from (12),

$$u_{n+r}^q = \mu\left\{xu_n^q + \left(q-1-\frac{n}{m}\right)u_n^{q-1}\right\} \text{ ..... (39),}$$

where $\mu = -\frac{bm}{a(m-r)}$.

And it is easy to see that we have generally

$$u^q_{n+qr} = \mu^q \{x^q u^q_n + A_1(n)\, x^{q-1} u^{q-1}_n + A_2(n)\, x^{q-2} u^{q-2}_n + \ldots$$
$$\ldots A_q(n)\, n(n-1) \ldots (n-q+1)\, u_n\} \ldots\ldots (40),$$

where $A_1(n)$, $A_2(n)$, &c. are functions of (n) to be determined, an equation which can obviously be depressed to the $(q-1)^{\text{th}}$ order when $n = 1, 2, 3 \ldots (q-1)$. It can also be shown that (40) admits of depression in the following cases, viz., $n = -1, -2, \ldots -q$.

7. The following process enables us to calculate with tolerable facility the value of $u^q_{n+(s+1)r}$ from u^q_{n+sr}, where s is less than q.

From equation (40) we have

$$u^q_{n+sr} = \mu^s \{x^s u^q_n + A_1(n)\, x^{s-1} u^{q-1}_n + A_2(n)\, x^{s-2} u^{q-2}_n$$
$$+ A_3(n)\, x^{s-3} u^{q-3}_n + \ldots A_s(n)\, u^{q-s}_n\} \ldots\ldots (41),$$

$$u^q_{n+(s+1)r} = \mu^{s+1} \{x^{s+1} u^q_n + B_1(n)\, x^s u^{q-1}_n + B_2(n)\, x^{s-1} u^{q-2}_n$$
$$+ B_3(n)\, x^{s-2} u^{q-3}_n + \ldots B_{s+1}(n)\, u^{q-s-1}_n\} \ldots\ldots (42)$$

$$= \mu^{s+1} \left\{ x^s \left[x u^q_n + \left(q-1-\frac{n}{m}\right) u^{q-1}_n \right] \right.$$
$$+ A_1(n+r)\, x^{s-1} \left[x u^{q-1}_n + \left(q-2-\frac{n}{m}\right) u^{q-2}_n \right]$$
$$\left. + A_2(n+r)\, x^{s-2} \left[x u^{q-2}_n + \left(q-3-\frac{n}{m}\right) u^{q-3}_n \right] + \text{&c.} \right\} \ldots (43).$$

Equate like powers of x in (42) and (43), then

$$\left.\begin{aligned}
B_1(n) &= A_1(n+r) + \left(q-1-\frac{n}{m}\right) \\
B_2(n) &= A_2(n+r) + \left(q-2-\frac{n}{m}\right) A_1(n+r) \\
B_3(n) &= A_3(n+r) + \left(q-3-\frac{n}{m}\right) A_2(n+r) \\
B_4(n) &= A_4(n+r) + \left(q-4-\frac{n}{m}\right) A_3(n+r) \\
B_5(n) &= A_5(n+r) + \left(q-5-\frac{n}{m}\right) A_4(n+r) \\
&\vdots \\
B_{s+1}(n) &= A_{s+1}(n+r) + \left(q-s-1-\frac{n}{m}\right) A_s(n+r)
\end{aligned}\right\} \ldots\ldots (44).$$

Hence the values of the functions $B_1(n)$, $B_2(n)$, &c., can be readily calculated from those of the functions $A_1(n)$, $A_2(n)$, &c.

The above equations will apply equally well to the calculation of $u^q_{n+(s+1)m}$ from u^q_{n+sm}, by changing r into m, and putting $\mu_1 = \dfrac{br}{m-r}$.

From (44) the functions $A_1(n+r)$, $A_2(n+r)$, &c. can be eliminated, and the resulting equation, involving the functions $B_1(n)$, $B_2(n)$, &c., will be found useful for purposes of verification.

8. Let us apply the process above indicated to the quintic

$$y^5+ay^4+bx=0.$$

Here $m=q=5$, $r=p=4$, and (39) becomes

$$u^5_{n+4}=-\frac{5b}{a}\left\{xu^5_n+\left(4-\frac{n}{5}\right)u^4_n\right\};$$

$$A_1(n+4)=\frac{16-n}{5},$$

$$\therefore\quad u^5_{n+8}=\frac{5^2b^2}{a^2}\left\{x^2u^5_n+\frac{36-2n}{5}xu^4_n+\frac{15-n}{5}\cdot\frac{16-n}{5}u^3_n\right\};$$

$$A_1(n+4)=\frac{28-2n}{5},\quad A_2(n+4)=\frac{11-n}{5}\cdot\frac{12-n}{5};$$

$$\therefore\quad u^5_{n+12}=-\frac{5^3b^3}{a^3}\left\{x^3u^5_n+\frac{48-3n}{5}\cdot x^2u^4_n\right.$$
$$+\left(\frac{11-n}{5}\cdot\frac{12-n}{5}+\frac{15-n}{5}\cdot\frac{28-2n}{5}\right)xu^3_n$$
$$\left.+\frac{10-n}{5}\cdot\frac{11-n}{5}\cdot\frac{12-n}{5}\cdot u^2_n\right\};$$

$$A_1(n+4)=\frac{36-3n}{5},\quad A_2(n+4)=\frac{7-n}{5}\cdot\frac{8-n}{5}+\frac{11-n}{5}\cdot\frac{20-2n}{5},$$
$$A_3(n+4)=\frac{6-n}{5}\cdot\frac{7-n}{5}\cdot\frac{8-n}{5},$$

$$\therefore\quad u^5_{n+16}=\frac{5^4b^4}{a^4}\left\{x^4u^5_n+\frac{56-4n}{5}\cdot x^3u^4_n\right.$$
$$+\left(\frac{7-n}{5}\cdot\frac{8-n}{5}+\frac{11-n}{5}\cdot\frac{20-2n}{5}+\frac{15-n}{5}\cdot\frac{36-3n}{5}\right)x^2u^3_n$$
$$+\left(\frac{6-n}{5}\cdot\frac{7-n}{5}\cdot\frac{8-n}{5}+\frac{10-n}{5}\cdot\frac{7-n}{5}\cdot\frac{8-n}{5}+\frac{10-n}{5}\cdot\frac{11-n}{5}\cdot\frac{20-2n}{5}\right)xu^2_n$$
$$\left.+\frac{5-n}{5}\cdot\frac{6-n}{5}\cdot\frac{7-n}{5}\cdot\frac{8-n}{5}u^1_n\right\};$$

$$A_1=\frac{40-4n}{5},\quad A_2(n+4)=\frac{3-n}{5}\cdot\frac{4-n}{5}+\frac{7-n}{5}\cdot\frac{12-2n}{5}+\frac{11-n}{5}\cdot\frac{24-3n}{5},$$
$$A_3(n+4)=\frac{3-n}{5}\cdot\frac{2(4-n)^2}{5^2}+\frac{7-n}{5}\cdot\frac{2(6-n)^2}{5^2},$$
$$A_4(n+4)=\frac{1-n}{5}\cdot\frac{2-n}{5}\cdot\frac{3-n}{5}\cdot\frac{4-n}{5},$$

$$\therefore \quad u^5_{n+20} = -\frac{5^5 b^5}{a^5}\left\{ x^5 u^5_n + (12-n)\, x^4 u^4_n \right.$$

$$+\left[\frac{3-n}{5}\cdot\frac{4-n}{5}+\frac{7-n}{5}\cdot\frac{12-2n}{5}+\frac{11-n}{5}\cdot\frac{24-3n}{5}+\frac{40-4n}{5}\cdot\frac{15-n}{5}\right] x^3 u^3_n$$

$$+\left[\frac{3-n}{5}\cdot\frac{2\,(4-n)^2}{5^2}+\frac{7-n}{5}\cdot\frac{2\,(6-n)^2}{5^2}\right.$$

$$\left.+\frac{10-n}{5}\left(\frac{3-n}{5}\cdot\frac{4-n}{5}+\frac{7-n}{5}\cdot\frac{12-2n}{5}+\frac{11-n}{5}\cdot\frac{24-3n}{5}\right)\right] x^2 u^2_n$$

$$+\left[\frac{1-n}{5}\cdot\frac{2-n}{5}\cdot\frac{3-n}{5}\cdot\frac{4-n}{5}\right.$$

$$\left.+\frac{5-n}{5}\left(\frac{3-n}{5}\cdot\frac{2\,(4-n)^2}{5}+\frac{7-n}{5}\cdot\frac{2\,(6-n)^2}{5^2}\right)\right] x u^1_n$$

$$\left.-\left[\frac{n}{5}\cdot\frac{1-n}{5}\cdot\frac{2-n}{5}\cdot\frac{3-n}{5}\cdot\frac{4-n}{5}\right] u_n \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (45).$$

Again, differentiate (13), and we have

$$u^5_{n+5} = 4b\left\{ x u^5_n + \left(4-\frac{n}{4}\right) u^4_n \right\};$$

$$A_1(n+5) = \frac{11-n}{4},$$

$$\therefore \quad u^5_{n+10} = 4^2 b^2 \left\{ x^2 u^5_n + \frac{27-2n}{4}\, x u_n + \frac{12-n}{4}\cdot\frac{11-n}{4}\, u^3_n \right\};$$

$$A_1(n+5) = \frac{17-n}{4}, \quad A_2(n+5) = \frac{7-n}{4}\cdot\frac{6-n}{4},$$

$$\therefore \quad u^5_{n+15} = 4^3 b^3 \left\{ x^3 u^5_n + \frac{33-3n}{4}\, x^2 u^4_n \right.$$

$$\left.+\left(\frac{7-n}{4}\cdot\frac{6-n}{4}+\frac{12-n}{4}\cdot\frac{17-2n}{4}\right) x u^3_n + \frac{8-n}{4}\cdot\frac{7-n}{4}\cdot\frac{6-n}{4}\cdot u^2_n \right\};$$

$$A_1(n+5) = \frac{18-3n}{4}, \quad A_2(n+5) = \frac{2-n}{4}\cdot\frac{1-n}{4}+\frac{7-n}{4}\cdot\frac{7-2n}{4},$$

$$A_3(n+5) = \frac{3-n}{4}\cdot\frac{2-n}{4}\cdot\frac{1-n}{4},$$

$$\therefore \quad u^5_{n+20} = 4^4 b^4 \left\{ x^4 u^5_n + \frac{34-4n}{4}\cdot x^3 u^4_n \right.$$

$$+\left[\frac{2-n}{4}\cdot\frac{1-n}{4}+\frac{7-n}{4}\cdot\frac{7-2n}{4}+\frac{12-n}{4}\cdot\frac{18-3n}{4}\right] x^2 u^3_n$$

$$\left.+\left[\frac{3-n}{4}\cdot\frac{2-n}{4}\cdot\frac{1-n}{4}+\frac{8-n}{4}\left(\frac{2-n}{4}\cdot\frac{1-n}{4}+\frac{7-n}{4}\cdot\frac{7-2n}{4}\right)\right] x u^2_n \right.$$

$$\left.+\left[\frac{4-n}{4}\cdot\frac{3-n}{4}\cdot\frac{2-n}{4}\cdot\frac{1-n}{4}\right] u^1_n \right. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (46).$$

Equating the values of u^5_{n+20} found in (45) and (46), we have a linear differential equation of the 5th order, which is satisfied by $c\beta_1^n$, where β_1 is a root of $y^5+ay^4+bx=0$.

9. To find the usual resolvent, take $n=1$, then (45) becomes

$$u^5_{21} = -\frac{5^5b^5}{a^5}\left\{x^4u_1^5 + 11x^3u_1^4 + \frac{156}{5}x^2u_1^3 + \frac{564}{5^2}xu_1^2 + \frac{1344}{5^4}u_1^1\right\}$$

$$= -\frac{5^5b^5}{a^5}\cdot\frac{d}{dx}\left\{x^4u_1^4 + 7x^3u_1^3 + \frac{51}{5}x^2u_1^2 + \frac{54}{5^2}xu_1^1 - \frac{6}{5^4}u_1\right\} \quad \ldots (47);$$

and (46) becomes

$$u^5_{21} = 4^4b^4\left\{x^3u_1^5 + \frac{15}{2}x^2u_1^4 + \frac{195}{4^2}xu_1^3 + \frac{210}{4^3}u_1^2\right\}$$

$$= 4^4b^4\frac{d}{dx}\left\{x^3u_1^4 + \frac{9}{2}x^2u_1^3 + \frac{51}{4^2}xu_1^2 + \frac{6}{4^3}u_1^1\right\} \quad \ldots\ldots\ldots (48).$$

Whence

$$x^3\left(4^4+\frac{5^5bx}{a^5}\right)u_1^4 + x^2\left(18\cdot4^3+\frac{7\cdot5^5bx}{a^5}\right)u_1^3 + x\left(51\cdot4^2+\frac{51\cdot5^4bx}{a^5}\right)u_1^2$$

$$+\left(24+\frac{54\cdot5^3bx}{a^5}\right)u_1^1 - \frac{30b}{a^5}u_1 = 0 = \frac{6b}{a^4} \quad \ldots\ldots\ldots (49).$$

This resolvent agrees with the form found by Mr. Harley in the "Manchester Memoirs," Vol. II., Third Series, page 241.

10. On the transformation of the differential resolvent when the algebraical equation is changed.

Mr. Harley, in the memoir above quoted (page 243), has remarked on the difficulty and importance of this transformation.

If we write $x^{-\frac{m}{m-r}}$ for x_1, and $x^{-\frac{1}{m-r}}\cdot y$ for y_1 ..... (50),

the equation $y_1^m + ay_1^r + bx_1 = 0$ ..... (51)

will be converted into $y^m + axy^r + b = 0$ ..... (52).

A particular form of this equation is discussed by Boole in the Transactions of the Royal Society, May 1864. In this paper, Boole remarks (page 738) that "not only for the particular case $n=-1$, but apparently for all integer values of n, the general differential equation admits of one integration. This statement requires, I think, reconsideration, as it does not seem to be confirmed by the foregoing investigation.

The resolvent of (52) is readily obtained from the resolvent of (51), by finding the values of $\frac{dy_1}{dx_1}$, $\frac{d^2y_1}{dx_1^2}$, &c. in terms of x, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., and substituting these values in the resolvent of (51).

In other words, change the variables in the resolvent of (51) by means of the substitutions (50), and the resulting equation will be the resolvent of (52).

Again, put $x_1 = x^{\frac{r}{m-r}}$, and $y_1 = x^{\frac{1}{m-r}}.\, y$ .....(53),

then the equation $y^m + ay^r + bx = 0$ .....(54)

will be changed into $xy^m + ay^r + b = 0$ .....(55).

Now change the variables in the resolvent of (54) by means of the relations (53), and the result will be the resolvent of (55).

Put $x_1 = x^{\frac{m-r}{r}}$, and $yy_1 = x^{\frac{1}{r}}$ .....(56),

then the equation $y_1^m + \frac{a}{b} y_1^{m-r} + \frac{1}{b} x = 0$ .....(57)

will be changed into $y^m + ay^r + bx = 0$ .....(58);

change the variables in the resolvent of (58) by means of the relations in (56), and the result will be the resolvent of (57).

11. Referring to the functions $B_1(n)$, $B_2(n)$ ... in equation (44), it will be seen that their calculation is made to depend upon the functions $A_1(n)$, $A_2(n)$ ... in (41), by means of the relations of (44). But Mr. Harley has communicated to me a rule by which the functions $B_1(n)$, $B_2(n)$ ..., &c. can be calculated at once without the assistance of the relations in (44).

Let $u_{n+r}^t = u\{xu_n^t + (t, n)\, u_n^{t-1}\}$,

where t is any positive integer, and

$$(t, n) = t - 1 - \frac{n}{m}.$$

Now, Mr. Harley has found that

$$u_{n+sr}^t = \mu^s \{x^s u_n^t + \pi_s(t, n)^1 x^{s-1} u_n^{t-1} + \pi_s(t-1, n)^2 x^{s-2} u_n^{s-2} + \pi_s(t-2, n)^3 x^{s-3} u_n^{t-3} + \&c.\},$$

where $\pi_s(n, t)^p$ represents the sum of all the functions which can be formed from

$$(n, t-p+1) \times (n+r, t-p+2) \times (n+2r, t-p+3) \ldots \times \{n+(p-1)r, t\},$$

by taking the several combinations of s terms of the arithmetical series n, $n+r$, $n+2r$, &c., p together. By means of this simple rule the whole of the foregoing results have been verified.

Addendum. By the Rev. Robert Harley, F.R.S.

The substance of the foregoing paper was communicated to me by the author several months ago, and I have since been led by the study of his results to some generalizations which it may be convenient to place on record here. I start from the equations (12) and (13), which Mr. Rawson obtains by means of Murphy's theorem, but which may be otherwise obtained by a very simple process suggested by Professor Cayley.

Write $\frac{b}{a}$ for a, and $\frac{c}{a}$ for b; then the equation

$$y^m + ay^r + bx = 0$$

becomes

$$ay^m + by^r + cx = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

which is unaltered when a, m are interchanged with b, r. Consider y as a function of x; then, differentiating (i.) with respect to x and multiplying the result by y^n, we obtain

$$(amy^{m+n-1} + bry^{r+n-1})\, y' + cy^n = 0;$$

but

$$my^{n-1}(ay^m + by^r + cx)\, y' = 0;$$

whence, eliminating y^{m+n-1},

$$b(m-r)\, y^{r+n-1}y' + cmxy^{n-1}y' - cy^n = 0;$$

or summing for any number (s) of the roots y,

$$b(m-r)\, \Sigma y^{r+n-1}y' + cmx\Sigma y^{n-1}y' - c\Sigma y^n = 0;$$

and since

$$u_n = -\frac{1}{n}\Sigma y^n,\ u'_n = -\Sigma y^{n-1}y',\ \&c.,$$

therefore

$$b(m-r)\, u'_{r+n} + cmxu'_n - cnu_n = 0,$$

or

$$u'_{r+n} = -\frac{c}{b}\left(\frac{m}{m-r}u'_n - \frac{n}{m-r}u_n\right) \quad \ldots\ldots\ldots\ldots\text{(ii.)}$$

Interchange a, m with b, r; then

$$u_{m+n} = -\frac{c}{a}\left(\frac{r}{r-m}u'_n - \frac{n}{r-m}u_n\right) \quad \ldots\ldots\ldots\ldots\text{(iii.)}$$

These results, viz. (ii.) and (iii.), agree with Mr. Rawson's equations (12) and (13). They may be otherwise written

$$Du_{r+n} = -\frac{c}{b}x\left(\frac{m}{m-r}D - \frac{n}{m-r}\right)u_n \quad \ldots\ldots\ldots\ldots\text{(iv.)},$$

$$Du_{m+n} = -\frac{c}{a}x\left(\frac{r}{r-m}D - \frac{n}{r-m}\right)u_n \quad \ldots\ldots\ldots\ldots\text{(v.)},$$

where $$D = x\frac{d}{dx}.$$

In what follows the usual factorial notation is employed, viz.,

$$[\theta]^{\alpha} = \theta\,(\theta-1)\,(\theta-2)\,\ldots\,(\theta-\alpha+1),$$

according to which $$[\theta]^{\alpha+\beta} = [\theta]^{\alpha}\,[\theta-\alpha]^{\beta},$$

and reductions are effected by the aid of the theorem

$$f(D)\,x^{\alpha}u = x^{\alpha}f(D+\alpha)\,u,$$

or $$x^{\alpha}f(D)\,u = f(D-\alpha)\,x^{\alpha}u.$$

See Boole's "Differential Equations," Chapters XVI. and XVII.

Operating with $[D-1]^{t-1}$ on both members of (iv.), we have

$$[D]^{t}u_{r+n} = -\frac{c}{b}\,[D-1]^{t-1}\,x\left(\frac{m}{m-r}D-\frac{n}{m-r}\right)u_n$$

$$= -\frac{c}{b}\,x\left(\frac{m}{m-r}D-\frac{n}{m-r}\right)[D]^{t-1}u_n.$$

Writing $r+n$ for n, this equation gives

$$[D]^{t}u_{2r+n} = -\frac{c}{b}\,x\left(\frac{m}{m-r}D-\frac{n+r}{m-r}\right)[D]^{t-1}u_{r+n}$$

$$= -\frac{c}{b}\,x\left(\frac{m}{m-r}D-\frac{n+r}{m-r}\right)\left(-\frac{c}{b}\,x\right)\left(\frac{m}{m-r}D-\frac{n}{m-r}\right)[D]^{t-2}u$$

$$= \left(-\frac{c}{b}\right)^{2}x^{2}\left[\frac{m}{m-r}D-\frac{n}{m-r}+1\right]^{2}[D]^{t-2}u_n.$$

Similarly,

$$[D]^{t}u_{3r+n} = \left(-\frac{c}{b}\right)^{3}x^{2}\left[\frac{m}{m-r}D-\frac{n+r}{m-1}+1\right]^{2}[D]^{t-2}u_{r+n}$$

$$= \left(-\frac{c}{b}\right)^{3}x^{3}\left[\frac{m}{m-r}D-\frac{n}{m-r}+2\right]^{3}[D]^{t-3}u_n.$$

And, in general,

$$[D]^{t}u_{qr+n} = \left(-\frac{c}{b}\right)^{q}x^{q}\left[\frac{m}{m-r}D-\frac{n}{m-r}+q-1\right]^{q}[D]^{t-q}u_n \quad \ldots\text{(vi.)}$$

By operating in the same way with (v.), or, more readily, by writing p for q, and interchanging a, m with b, r, we find

$$[D]^{t}u_{pm+n} = \left(-\frac{c}{a}\right)^{p}x^{p}\left[\frac{r}{r-m}D-\frac{n}{r-m}+p-1\right]^{p}[D]^{t-p}u_n$$

$$= \left(\frac{c}{a}\right)^{p}x^{p}\left[\frac{r}{m-r}D-\frac{n}{m-r}\right]^{p}[D]^{t-p}u_n \quad \ldots\ldots\ldots\ldots\text{(vii.)}$$

Make $pm = qr$; then

$$[D]^t u_{pm+n} = [D]^t u_{qr+n},$$

and (vi.), (vii.) give

$$\left[\frac{r}{m-r}D - \frac{n}{m-r}\right]^p [D]^{t-p} u_n$$

$$= (-)^q \frac{a^p c^q}{b^q c^p} x^{q-p} \left[\frac{m}{m-r}D - \frac{n}{m-r} + q - 1\right]^q [D]^{t-q} u_n \quad \text{.....(viii.)}$$

Suppose $m > r$, and therefore $q > p$; then, putting $t = q$, and interpreting $[D]^0$ by 1, we have

$$\left[\frac{r}{m-r}D - \frac{n}{m-r}\right]^p [D]^{q-p} u_n$$

$$= (-)^q \frac{a^p c^q}{b^q c^p} x^{q-p} \left[\frac{m}{m-r}D - \frac{n}{m-r} + q - 1\right]^q u_n$$

$$= (-)^q \frac{a^p c^q}{b^q c^p} \left[\frac{m}{m-r}D - \frac{n}{m-r} - 1\right]^q x^{q-p} u_n \quad \text{.....(ix.)}$$

Whence, writing m for q and r for p, and restoring the value of D, viz. $x\frac{d}{dx}$, we obtain

$$\left[\frac{r}{m-r}x\frac{d}{dx} - \frac{n}{m-r}\right]^r \left[x\frac{d}{dx}\right]^{m-r} u_n$$

$$= (-)^m \frac{a^r c^m}{b^m c^r} \left[\frac{m}{m-r}x\frac{d}{dx} - \frac{n}{m-r} - 1\right]^m x^{m-r} u_n \quad \text{.....(x.)},$$

the general integral of which is

$$u_n = C_1 y_1^n + C_2 y_2^n \ldots + C_m y_m^n.$$

On this result Professor Cayley remarks:—"Suppose $r > m$, say $= m + \theta$, there are r roots, and an integral will be

$$u_n = C_1 y_1^n + C_2 y_2^n \ldots + C_r y_r^n,$$

containing r constants. But in this case $[D]^{m-r} u_n = [D]^{-\theta} u_n$ is an inverse expression containing θ arbitrary constants, which remain in the differential equation: there should be m constants of integration, and thus in all $\theta + m_1 = r$ constants; viz., it is only on the supposition that the θ constants are arbitrary, that the integral contains the r arbitrary constants $C_1, C_2 \ldots C_r$. Or, more clearly, thus,—the differential equation of order m contains θ arbitrary constants; and to get rid of these we must pass to a differential equation of order $m + \theta_1 = r$; this is, in fact, the differential equation of the order r given by the interchange of m, a with r, b; and the integral of this is

$$u_n = C_1 y_1^n \ldots + C_r y_r^n,$$

containing, as it should do, r constants. This is very pretty indeed."

Other considerations lead to the same conclusion. For, if in (viii.) we put $t = p$, and write, as before, m for q and r for p, we have

$$\left[\frac{r}{m-r}D-\frac{n}{m-r}\right]^r u_n$$

$$=(-)^m\frac{a^r c^m}{b^m c^r}x^{m-r}\left[\frac{m}{m-r}D-\frac{n}{m-r}+m-1\right]^m [D]^{r-m}u_n,$$

which, since

$$\left[\frac{r}{m-r}D-\frac{n}{m-r}\right]^r=(-)^r\left[\frac{r}{r-m}D-\frac{n}{r-m}+r-1\right]^r,$$

and $$\left[\frac{m}{m-r}D-\frac{n}{m-r}+m-1\right]^m=(-)^m\left[\frac{m}{r-m}D-\frac{n}{r-m}\right]^m,$$

may be written thus

$$(-)^r x^{r-m}\left[\frac{r}{r-m}D-\frac{n}{r-m}+r-1\right]^r u_n$$

$$=\frac{a^r c^m}{b^m c^r}\left[\frac{m}{r-m}D-\frac{n}{r-m}\right]^m [D]^{r-m}u_n,$$

or thus

$$\left[\frac{m}{r-m}x\frac{d}{dx}-\frac{n}{r-m}\right]^m\left[x\frac{d}{dx}\right]^{r-m}u_n$$

$$=(-)^r\frac{b^m c^r}{a^r c^m}\left[\frac{r}{r-m}x\frac{d}{dx}-\frac{n}{r-m}-1\right]^r x^{r-m}u_n \quad \text{.....(xi.),}$$

an equation of the order r, which becomes identical with (x.) by the interchange of m, a with r, b.

The substitution $\begin{pmatrix} r, & m, & b \\ -r, & m-r, & c \end{pmatrix}$ changes the algebraical equation (i.) into $$ay^m+bxy^r+c=0 \quad \text{.....(xii.),}$$ and the same substitution changes the differential equation (x.) into

$$\left[-\frac{r}{m}x\frac{d}{dx}-\frac{n}{m}\right]^{-r}\left[x\frac{d}{dx}\right]^m u_n$$

$$=(-)^{m-r}\frac{b^m c^r}{a^r c^m}\left[\frac{m-r}{m}x\frac{d}{dx}-\frac{n}{m}-1\right]^{m-r}x^m u_n,$$

which, since $$[-\theta]^{-a}=\frac{1}{[-\theta+a]^a}=\frac{(-1)^a}{[\theta-1]^a},$$

is equivalent to

$$\left[x\frac{d}{dx}\right]^m u_n=(-)^m\frac{b^m c^r}{a^r c^m}\left[\frac{r}{m}x\frac{d}{dx}+\frac{n}{m}-1\right]^r$$

$$\times\left[\frac{m-r}{m}x\frac{d}{dx}-\frac{n}{m}-1\right]^{m-r}x^m u_n \quad \text{.....(xiii.)}$$

This confirms a result which I obtained some years ago by the aid of Lagrange's theorem, viz., that the differential equation

$$a^r \left[x\frac{d}{dx}\right]^n u = \left[\frac{n-r}{n}x\frac{d}{dx}+\frac{m}{n}-1\right]^{n-r}\left[\frac{r}{n}x\frac{d}{dx}-\frac{m}{n}-1\right]^r x^{n-r}u$$

is satisfied by the m^{th} power of any root of the equation

$$y^n - xy^{n-r} + a = 0.$$

See Boole's "Differential Equations," Supplementary volume, p. 199.* See also "Proceedings of the London Mathematical Society, Vol. I., Paper IV.

In certain cases the differential equations (x), (xi.), and (xiii.) admit of reduction to equations of lower orders. And, in particular, this happens when m and r are not prime to each other. For if $m = a\mu$, and $r = a\rho$, a being the greatest common measure between m and r, the condition $pm = qr$ is satisfied by making $p = \rho$, and $q = \mu$; then (ix.) becomes

$$\left[\frac{\rho}{\mu-\rho}x\frac{d}{dx}-\frac{n}{a(\mu-\rho)}\right]^\rho\left[x\frac{d}{dx}\right]^{\mu-\rho}u_n$$

$$= (-)^\mu\frac{a^\rho c^\mu}{b^\mu c^\rho}\left[\frac{\mu}{\mu-\rho}x\frac{d}{dx}-\frac{n}{a(\mu-\rho)}-1\right]^\mu x^{\mu-\rho}u_n \quad \ldots(\text{xii.}),$$

and, by the interchange of a, μ with b, ρ,

$$\left[\frac{\mu}{\rho-\mu}x\frac{d}{dx}-\frac{n}{a(\rho-\mu)}\right]^\mu\left[x\frac{d}{dx}\right]^{\rho-\mu}u_n$$

$$= (-)^\rho\frac{b^\mu c^\rho}{a^\rho c^\mu}\left[\frac{\rho}{\rho-\mu}x\frac{d}{dx}-\frac{n}{a(\rho-\mu)}-1\right]^\rho x^{\rho-\mu}u_n \quad \ldots(\text{xv.}),$$

which are both satisfied by the n^{th} power of any root of the algebraic equation (i.) And, in like manner, the differential equation

$$\left[x\frac{d}{dx}\right]^\mu u_n = (-)^\mu\frac{b^\mu c^\rho}{a^\rho c^\mu}\left[\frac{\rho}{\mu}x\frac{d}{dx}+\frac{n}{a\mu}-1\right]^\rho$$

$$\times\left[\frac{\mu-\rho}{\mu}x\frac{d}{dx}-\frac{n}{a\mu}-1\right]^{\mu-\rho}x^\mu u_n \quad \ldots\ldots\ldots\ldots(\text{xvi.})$$

is satisfied by the n^{th} power of any root of the algebraic equation (xii.). Equations (xiv.), (xv.), and (xvi.) are obviously reduced forms of (x.), (xi.), and (xiii.) respectively.

Other reductions may also be effected for particular values of n; notably in the case $n = 1$, when each differential equation may be

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* A misprint occurs in the seventh line from the top of the page, where the factor x^r should be x^{n-r}.

reduced to an order lower by unity than itself. But I will not pursue the subject any farther here.

The passage from the symbolical to the ordinary form of differential equations presents no difficulty.

Notes on the Geometric Solution of some Statical Problems connected with Mechanisms (*Linkworks*). *By* Alexander B. W. Kennedy, Professor of Engineering in University College, London.

[*Read May 9th*, 1878.]

The principal object of the following notes is to point out an exceedingly simple, and it is believed new, solution of the following general problem: Given any plane mechanism or linkwork, and a force acting upon any one of its links, to find by a construction the force which will balance the mechanism if acting in a given direction upon any other link. In examining this problem, it will be convenient, in the first instance, to look at a few very simple cases, in order to be able the more clearly to point out the identity of the solution in these and in the most complex ones.

In what follows I shall use the following notation:—small letters, a, b, c &c., for the links of a mechanism and capital letters for its vertices; the letter O always for an instantaneous centre, with suffixes denoting the links for whose relative motion it is the centre,—thus O_{ab} is the instantaneous centre for the relative motion of the links a and b; forces will be denoted by the letter f, with a suffix corresponding to the link on which the force is supposed to act. In the figures the link which is supposed to be fixed is distinguished in each case by shading.

Let there be supposed given, first, the simple linkwork $abcd$ (Fig. 1) of four bars, of which the link d is fixed. The force f_a is also given and the direction of f_c,—the problem being to find f_c. In the simplest possible case—a very common one, however,—where f_a passes through O_{ab} and f_c through O_{cb}, (that is, where the points of a and c on which the forces act are both also points of the same link b,) the easiest construction is of course to suppose them both acting on b, and resolve them through their join and through O_{bd}, as we know that their resultant must pass through that point. In the general case, however, as in Fig. 1, where f_a and f_c do *not* pass

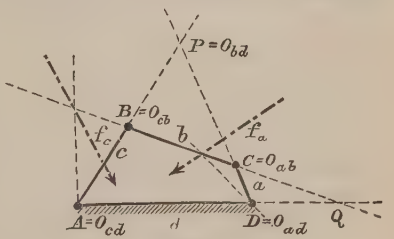

Fig. 1.

through O_{ab} and O_{cb}, another construction is required. There is no difficulty in resolving f_a in the direction of the axis of b and the line (dotted in figure) joining $\overline{f_a b}$ with the point O_{ad}, and then resolving the component along b (the only one which requires to be balanced) along the given direction f_c and through the join of $\overline{f_c b}$ and O_{cd}. But this plan, although convenient enough, is indirect, and moreover can seldom be applied to complex mechanisms at all. A really direct and general solution can, however, readily be found, by which the problem can be reduced to the conditions mentioned above as the simplest possible. We shall first describe this solution, and then point out an equally general form, somewhat less direct, but simpler, and for practical purposes in general much more convenient.

The mechanism *abcd* would remain unchanged if we could substitute for any of its links others moving about the same instantaneous centres and having the same relative velocities as those actually belonging to the mechanism. It will be convenient for us to make this change for links a and c, upon which the forces are supposed to act. Although we cannot, in the general case, make any such substitution as this *permanently*; yet for any instant it is possible to find an infinite number of mechanisms such that, with entirely different *lengths* of a and c, these links still remain *kinematically* unchanged, that is, they still move with the same relative velocities and about the same points.

Considering the figure *abcd* (Fig. 2) merely as a four-side, of which A, B, C, D, P and Q are the vertices, we know that

$$\frac{QA}{QD} = \frac{PC}{PB} \cdot \frac{AB}{DC}.$$

But, taking it as a linkwork, $P = O_{bd}$ and $Q = O_{ac}$; and probably the easiest proof of the above equality is that each ratio is equal to the ratio $\frac{\angle^r \text{ vel. } a}{\angle^r \text{ vel. } c}$. If now

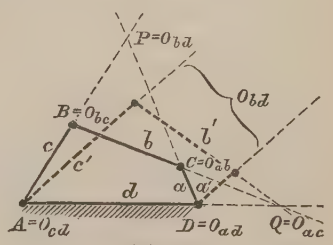

Fig. 2.

for a and c we substitute *any* two links a' and c' parallel to each other, the point $O_{bd} = P$ goes to ∞ and $\frac{PC'}{PB'} = 1$. To make, therefore, the ratio $\frac{\angle^r \text{ vel. } a'}{\angle^r \text{ vel. } c'} = \frac{\angle^r \text{ vel. } a}{\angle^r \text{ vel. } c}$, we must have $\frac{QA}{QD} = \frac{AB'}{DC'}$, which is obtained at once by taking for the axis of the new link b' either the existing axis of b or any other line passing through $O_{ac} = Q$. In a new mechanism, therefore, proportioned as $a'b'c'd$ (Fig. 2), the links a' and c' on which the forces act have the same velocity ratio as the original links a and c, and move about the same points ($O_{cd} = A$ and $O_{ad} = D$), and may therefore be substituted for them either in kinematic or static problems. I propose to call a substituted mechanism or link-

work of this kind a "virtual mechanism." The parallelism of the links a' and c' is not, of course, an essential condition of the virtual mechanism; but this form is probably simpler than any more general one, and fulfils all the necessary conditions. In order, now, to bring the case to the simplest possible form already illustrated, we must find, out of the infinite number of possible virtual mechanisms, the particular one which fulfils the condition that

$$O_{a'b'} \text{ lies upon } f_a,$$
$$O_{c'b'} \text{ lies upon } f_c,$$

while still $\overline{O_{c'b'} \cdot O_{a'b'}}$ (or b') passes through O_{ac}, and $\overline{O_{c'd'} O_{c'b'}}$ (or c') is parallel to $\overline{O_{a'd'} O_{a'b'}}$ (or a'). Geometrically this is easy enough. O_{ad} and O_{cd} (Fig. 3) are the centres of pencils, which being parallel are projective and determine in the lines f_a and f_c two projective rows. Corresponding points in these rows are the points simply in which the lines are cut by a' and c'. As the joins of these points envelope a conic, and as the link b' is to be one of such joins, the problem resolves itself simply into finding the particular tangent to the conic which passes through O_{ac}. There is only one solution, as the second tangent already exists in the line d, which is not available for our purposes.

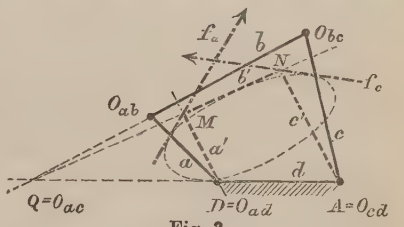

Fig. 3.

In this way the virtual mechanism can be drawn directly in its most convenient form, and the problem solved by the mere construction of a triangle of forces of which the third side is parallel to a' or c', *i.e.*, passes through $O_{b'd}$ at ∞. Two or three special cases are worth mentioning :—

(i.) If either force pass through the instantaneous centre of its own link relatively to the fixed one (as the point O_{ad} for a or O_{cd} for c), the problem is fictitious; as is shown at once by the conic disappearing, one of the pencils determining a point only, and not a row.

(ii.) If one force pass through O_{ac}, its own direction is that of the required tangent, and the virtual mechanism can be drawn at once without the aid of the conic.

(iii.) If *both* forces pass through O_{ac}, the conic becomes a point, and the mechanism closes up into a line, a' coinciding with DQ, and c' with AQ, and O_{cd} being the point at ∞ on the line d.

Although this solution is quite simple geometrically, and in its proof, it is yet somewhat complex from a constructive point of view, and moreover is unsuitable for use in elementary teaching, before the

theory of conics has been touched. For practical purposes, therefore, that is, to save drawing, as well as in the more elementary treatment of such problems, the following method will probably be more useful. It is little less direct than the one just given.

Take the link a' coinciding with a (Fig. 4) and determined in length by the point M at which f_a cuts a. Through this point and O_{ac} draw b', and then draw c' parallel to a'. The virtual mechanism thus becomes $DMNA$, the one force f_a acting through one of its vertices M. The link b' has again its instantaneous centre relatively to d at infinity; it has therefore a motion of translation only, and is in consequence *in equilibrio* under any number of couples.* We may consequently shift f_a parallel to itself to N, as f'_a, without altering the condition of the mechanism. To solve the problem, then, we have simply to take the equilibrium of the link c' under f_c, f'_a and their resultant, which is known in direction, as it must pass through the point O_{cd} as well as through the join of f_c and f'_a.

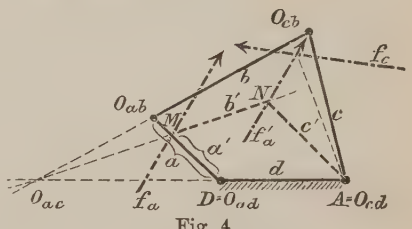

Fig. 4.

The whole of the foregoing applies without alteration to a mechanism of any degree of complexity, and renders the solution of statical problems, once the various instantaneous centres are known, equally simple in all cases. One illustration will suffice. Let us take the Peaucellier linkwork, Fig. 5, using the same lettering as before. We have first to find the points O_{ad}, O_{cd}, and O_{ac}. Of these we have the first directly as the point D of the fixed link. Such a point as O_{cd}, the instantaneous centre for the motion of c relatively to d, is generally most easily determined by finding the motion of two points in c and taking the join of the normals to the directions of their motion. Or, as it may be put more shortly, by taking the join of the in-

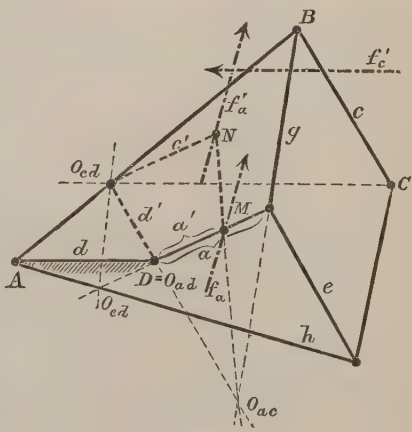

Fig. 5.

* The use of the instantaneous centre makes the proof of this very simple :—a body is *in equilibrio* under all forces passing through its instantaneous centre; but, taking a

stantaneous radii of any two points of c for which these are known. In this case the point B of c moves about A, and the point C describes a known line, the normal to which is parallel to d: these conditions give us O_{cd} at once. To make the case more general, however, it may be worth while pointing out that we do not require to assume the motion of C as known. If it were not, we should not know directly the motion of any other point than B in c. But c has one point in common with every other link in the mechanism,—its instantaneous centre relatively to that link,—and this point has the same motion to whichever of its pair of links we consider it to belong. This always gives us sufficient command of point-motions to find the centre that we want. In this case, for example, we might take instead of C, the point which is the instantaneous centre of c relative to e, obviously the point at ∞ on the direction of g. This point as a point of e is moving about O_{ed} (which we have at once as the join of a and h), and therefore has the same motion as a point of c. The normal to its motion we get, therefore, by drawing through O_{ed} a line parallel to g, which, of course, cuts AB in the point O_{cd} already mentioned. It remains to find O_{ac}. This point must be on the line joining O_{ad} and O_{cd},* and is fully determined therefore by the join of that line with g, the normal to the motion of the point B of c relatively to a.

The instantaneous centres being thus found, the number of links in the mechanism no longer affects the question,—we substitute c' and a' for c and a exactly as before, suppose f_a shifted to N, as f'_a, and for the required force f_c resolve f'_a in the directions of f_c and of the line joining $\overline{f_c f'_a}$ with O_{cd}.

It is not necessary to give further illustrations of this method of treating these problems; I have not found any cases, however apparently complex, in which it does not work just as easily as in the one given. It is worth while mentioning that this treatment of mechanisms, by the method of instantaneous centres, gives very great facilities for handling kinetic as well as static problems relating to them by purely geometric (graphic) methods.

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couple as equivalent to an infinitely small force acting along the line at infinity, it passes through all points at infinity, and therefore through the instantaneous centre of any body having a motion of pure translation.

* If a, b, and c be any three bodies having plane motion, their three instantaneous centres O_{ab}, O_{bc}, and O_{ca} are three points in a line.

On a Method in the analysis of Plane Curves. *By* J. J. Walker.

[*Read June* 13*th*, 1878.]

I have been led to attempt the development of a method of treating the intersections of a transversal with a plane curve, which occurred to me some years since, and which appears to me in many cases to offer facilities in the discussion of general properties not presented by other recognised methods; as well as to be calculated to introduce more uniformity of procedure in the treatment of questions to which it is applicable.

In what follows will be found:—

I. An introduction of the method, leading to the consideration of a certain ternary operator D:

II. The formation of the results of operating repeatedly with D on any ternary form, carried as far as the fourth operation; also,

III. That of the product of the results of operating with D on each of two forms:

IV. The deduction of the results of operating with a special form of D, indicated by D', which agree with some previously known:

V. A discussion, by the use of the method, of the problem of the Inflexion-Tangential Curve for the quartic; preceded by the determination of the coordinates of the tangential point in terms of the coordinates of the corresponding point of inflexion.

The results arrived at in this last division of the memoir are, as far as I am aware, all new.

I. *Introductory.*

If θ_1, θ_2, θ_3 are the angles which the transversal

$$lx+my+nz=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

makes with the sides BC, CA, AB, respectively, of the triangle of reference,

$$\left.\begin{aligned}
\sin\theta_1 &= \frac{m\sin C-n\sin B}{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}},\\
\sin\theta_2 &= \frac{n\sin A-l\sin C}{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}},\\
\sin\theta_3 &= \frac{l\sin B-m\sin A}{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}},
\end{aligned}\right\} \ldots (2),$$

the angles θ_1, θ_2, θ_3 being understood as those through which the transversal must rotate about a fixed point, O, in its length, so that a given segment OO' may coincide successively with lines drawn from O in the directions BC, CA, AB respectively. If, further, ρ' is the length of a segment of (1), or any parallel to (1), terminated by points O, O', the coordinates of which are xyz, $x'y'z'$, then

$$\frac{x-x'}{\rho'} = \sin\theta_1, \quad \frac{y-y'}{\rho'} = \sin\theta_2, \quad \frac{z-z'}{\rho'} = \sin\theta_3,$$

whence, writing α, β, γ for $\sin A, \sin B, \sin C$, if

$$\rho = \frac{\rho'}{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}},$$

$$x' = x+(m\gamma-n\beta)\rho, \quad y' = y+(n\alpha-l\gamma)\rho, \quad z' = z+(l\beta-m\alpha)\rho \ldots (3).$$

Let $f(x'y'z') = 0$ be the equation to any plane curve of order p, and $O_1, O_2 \ldots$ the points in which it is met by the transversal (1); the lengths $\rho_1', \rho_2' \ldots$ of the segments $OO_1, OO_2 \ldots$ will be determined by the equation

$$f(xyz)+p\rho Df(xyz)+\frac{p\,.\,p-1}{1\,.\,2}\rho^2D^2f(xyz)+\ldots+\rho^pD^pf(xyz) = 0 \ldots (4),$$

where the operator

$$(p\,.\,p-1\ldots p-r+1)\,D^r = \left\{(m\gamma-n\beta)\frac{d}{dx}+(n\alpha-l\gamma)\frac{d}{dy}+(l\beta-m\alpha)\frac{d}{dz}\right\}^r \ldots\ldots (5);$$

and if $x'y'z'$ are the coordinates of one of the points $O_1 \ldots O_p$, the values of x' are given by

$$f(xyz)+p\frac{x'-x}{m\gamma-n\beta}Df(xyz)+\frac{p\,.\,p-1}{1\,.\,2}\left(\frac{x'-x}{m\gamma-n\beta}\right)^2D^2f(xyz)+\ldots$$
$$\ldots+\left(\frac{x'-x}{m\gamma-n\beta}\right)^pD^pf(xyz) = 0 \ldots\ldots (6),$$

while y', z' are determined by analogous equations.

II. *Formation of* $D^2v \ldots D^4v$.

By definition—v being any ternary form of the order q—

$$xyz\,q\,(q-1)\,D^2v = (m\gamma-n\beta)^2xyz\frac{d^2v}{dx^2}+(n\alpha-l\gamma)^2xyz\frac{d^2v}{dy^2}+(l\beta-m\alpha)^2xyz\frac{d^2v}{dz^2}$$
$$+2(n\alpha-l\gamma)(l\beta-m\alpha)xyz\frac{d^2v}{dy\,dz}+2(l\beta-m\alpha)(m\gamma-n\beta)xyz\frac{d^2v}{dz\,dx}$$
$$+2(m\gamma-n\beta)(n\alpha-l\gamma)xyz\frac{d^2v}{dx\,dy} \ldots\ldots (7);$$

but, by the properties of homogeneous functions,

$$2yz\frac{d^2v}{dy\,dz} = (q-1)\left(-x\frac{dv}{dx}+y\frac{dv}{dy}+z\frac{dv}{dz}\right)+x^2\frac{d^2v}{dx^2}-y^2\frac{d^2v}{dy^2}-z^2\frac{d^2v}{dz^2},$$

$$2zx\frac{d^2v}{dz\,dx} = (q-1)\left(x\frac{dv}{dx}-y\frac{dv}{dy}+z\frac{dv}{dz}\right)-x^2\frac{d^2v}{dx^2}+y^2\frac{d^2v}{dy^2}-z^2\frac{d^2v}{dz},$$

$$2xy\frac{d^2v}{dx\,dy} = (q-1)\left(x\frac{dv}{dx}+y\frac{dv}{dy}-z\frac{dv}{dz}\right)-x^2\frac{d^2v}{dx^2}-y^2\frac{d^2v}{dy^2}+z^2\frac{d^2v}{dz^2}$$

and eliminating $\frac{d^2v}{dy\,dz}$, $\frac{d^2v}{dz\,dx}$, $\frac{d^2v}{dx\,dy}$, by these substitutions, from (7),

$$xyz\,q\,(q-1)\,D^2v = (m\gamma-n\beta)^2\,xyz\frac{d^2v}{dx^2}+(n\alpha-l\gamma)^2\,xyz\frac{d^2v}{dy^2}+(l\beta-m\alpha)^2\,xyz\frac{d^2v}{dz^2}$$

$$+(n\alpha-l\gamma)(l\beta-m\alpha)\,x\left\{(q-1)\left(-x\frac{dv}{dx}+y\frac{dv}{dy}+z\frac{dv}{dz}\right)+x^2\frac{d^2v}{dx^2}-y^2\frac{d^2v}{dy^2}-z^2\frac{d^2v}{dz^2}\right\}$$

$$+(l\beta-m\alpha)(m\gamma-n\beta)\,y\left\{(q-1)\left(\ x\frac{dv}{dx}-y\frac{dv}{dy}+z\frac{dv}{dz}\right)-x^2\frac{d^2v}{dx^2}+y^2\frac{d^2v}{dy^2}-z^2\frac{d^2v}{dz^2}\right\}$$

$$+(m\gamma-n\beta)(n\alpha-l\gamma)\,z\left\{(q-1)\left(\ x\frac{dv}{dx}+y\frac{dv}{dy}-z\frac{dv}{dz}\right)-x^2\frac{d^2v}{dx^2}-y^2\frac{d^2v}{dy^2}+z^2\frac{d^2v}{dz^2}\right\},$$

in which the coefficient of α^2 =

$$-mnx^3\frac{d^2v}{dx^2}+nxy\,(my+nz)\frac{d^2v}{dy^2}+mzx\,(my+nz)\frac{d^2v}{dz^2}-(q-1)mnx\left(-x\frac{dv}{dx}+y\frac{dv}{dy}+z\frac{dv}{dz}\right),$$

or $\left(\text{since } my+nz=-lx,\ -x\frac{dv}{dx}+y\frac{dv}{dy}+z\frac{dv}{dz}=-2x\frac{dv}{dx}+qv\right)$

$$-x^2\left(mnx\frac{d^2v}{dx^2}+nly\frac{d^2v}{dy^2}+lmz\frac{d^2v}{dz^2}\right)+2\,(q-1)\,mnx^2\frac{dv}{dx}-q\,(q-1)\,mnxv.$$

The coefficient of $\beta\gamma$ in the same is, similarly, equal to

$$-2yz\left(mn\frac{d^2v}{dx^2}+nly\frac{d^2v}{dy^2}+lmz\frac{d^2v}{dz^2}\right)+2\,(q-1)\,yz\left(nl\frac{dv}{dy}+lm\frac{dv}{dz}\right).$$

Hence it appears at once that, if $\alpha x+\beta y+\gamma z=\Delta$,

$$xyz\,q\,(q-1)\,D^2v = -\Delta^2\left(mnx\frac{d^2v}{dx^2}+nly\frac{d^2v}{dy^2}+lmz\frac{d^2v}{dz^2}\right)$$
$$+2\,(q-1)\,\Delta\left(mn\alpha x\frac{dv}{dx}+nl\beta y\frac{dv}{dy}+lm\gamma z\frac{dv}{dz}\right)$$
$$-q\,(q-1)(mn\alpha^2x+nl\beta^2y+lm\gamma^2z)\,v\ldots\ldots\ldots\ldots(8).$$

If $$mnx=\lambda,\quad nly=\mu,\quad lmz=\nu,$$
$$R=\alpha^2\lambda+\beta^2\mu+\gamma^2\nu,$$

and if P, Q are operators defined thus—the subject being of order q—

viz., $$q\,(q-1)\,\ldots\,(q-2r+1)\,P^r=\left(\lambda\frac{d^2}{dx^2}+\mu\frac{d^2}{dy^2}+\nu\frac{d^2}{dz^2}\right)^r,$$

$$q\,(q-1)\,\ldots\,(q-r+1)\,Q^r=\left(\alpha\lambda\frac{d}{dx}+\beta\mu\frac{d}{dy}+\gamma z\frac{d}{dz}\right)^r,$$

where λ, μ, ν are to be regarded as constants in performing the differentiations, and replaced by their values $mnx\ldots$ only after differentiating,

then $$xyzD^2v=(-\Delta^2P+2\Delta Q-R)\,v\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

To form D^3v:

$$xyz\,q\,(q-1)(q-2)\,D^3v = xyz\,(q-1)(q-2)\,D^2\,(qDv)$$

$$= \{-\Delta^2\,(q-1)(q-2)\,P + 2\Delta\,(q-2)(q-1)\,Q - (q-1)(q-2)\,R\}$$
$$\times \left\{(m\gamma - n\beta)\frac{dv}{dx} + (n\alpha - l\gamma)\frac{dv}{dy} + (l\beta - m\alpha)\frac{dv}{dz}\right\}$$

$$= \left\{-\Delta^2\left(\lambda\frac{d^2}{dx^2} + \mu\frac{d^2}{dy^2} + \nu\frac{d^2}{dz^2}\right) + 2\,(q-2)\,\Delta\left(\lambda\alpha\frac{d}{dx} + \mu\beta\frac{d}{dy} + \nu\gamma\frac{d}{dz}\right)\right.$$
$$\left. - (q-1)(q-2)(\lambda\alpha^2 + \mu\beta^2 + \nu\gamma^2)\right\}\left\{(m\gamma - n\beta)\frac{dv}{dx} + \ldots\ldots\right\}$$

$$= -\Delta^2\left\{(m\gamma - n\beta)\,mnx\frac{d^3v}{dx^3} + (n\alpha - l\gamma)\,nly\frac{d^3v}{dy^3} + (l\beta - m\gamma)\,lmz\frac{d^3v}{dz^3}\right.$$
$$+ (n\alpha - l\gamma)\,mnx\frac{d^3v}{dx^2\,dy} + (l\beta - m\alpha)\,lmnx\frac{d^3v}{dx^2\,dz} + (m\gamma - n\beta)\,nly\frac{d^3v}{dy^2\,dx}$$
$$\left. + (l\beta - m\alpha)\,nly\frac{d^3v}{dy^2\,dz} + (m\gamma - n\beta)\,lmz\frac{d^3v}{dz^2\,dx} + (n\alpha - l\gamma)\,lmz\frac{d^3v}{dz^2\,dy}\right\}$$
$$+ 2\,(q-2)\,\Delta\left[(m\gamma - n\beta)\,mnax\frac{d^2v}{dx^2} + (n\alpha - l\gamma)\,nl\beta y\frac{d^2v}{dy^2}\right.$$
$$\left. + (l\beta - m\alpha)\,lm\gamma z\frac{d^2v}{dz^2} + \{(n\alpha - l\gamma)\,lm\gamma z + (l\beta - m\alpha)\,nl\beta y\}\frac{d^2v}{dy\,dz} + \ldots\frac{d^2v}{dx\,dy}\right]$$
$$- (q-1)(q-2)(mn\alpha^2 x + nl\beta^2 y + lm\gamma^2 z)\left\{(m\gamma - n\beta)\frac{dv}{dx} + \ldots\ldots\right\}.$$

But I give this development rather to illustrate the use of the operators P, Q, than for any necessity that will arise for employing it, as, in practice, any odd power of Dv is multiplied by some other odd power; and the development of the product is much more easily effected by methods to be subsequently treated of than by forming the product of the developments.

To form D^4v:

$$xyzD^4v = (-\Delta^2 P + 2\Delta Q - R)\,D^2v = -\Delta^2 D^2\,(Pv) + 2\Delta D^2\,(Qv) - RD^2v,$$

$$x^2y^2z^2D^4v = (-\Delta^2 P + 2\Delta Q - R)^2\,v$$
$$= \Delta^4 P^2 v - 4\Delta^3 PQv + \Delta^2\,(4Q^2 + 2RP)\,v - 4\Delta RQv + R^2v \ldots (10).$$

In what follows I shall not make use of any further developments, but the methods of forming D^5v, D^6v ... are evident; viz.,

$$x^2y^2z^2D^5u = (-\Delta^2 P + 2\Delta Q - R)^2\,Dv,$$

and
$$x^3y^3z^3D^6v = (-\Delta^2 P + 2\Delta Q - R)^3\,v.$$

In the application of these developments, it will be found very con-

venient to write—u being of order p—u', u'', u''' for $\frac{1}{p}\frac{du}{dx}$, $\frac{1}{p}\frac{du}{dy}$, $\frac{1}{p}\frac{du}{dz}$, also a, b, c, f, g, h for $\frac{1}{p\,(p-1)}\frac{d^2u}{dx^2}$ ..... $\frac{1}{p\,(p-1)}\frac{d^2u}{dx\,dy}$; thus

$$Du = (m\gamma - n\beta)\,u' + (n\alpha - l\gamma)\,u'' + (l\beta - m\alpha)\,u''',$$

$$D^2u = -\Delta^2\,(mnx\alpha + nly\beta + lmz\gamma) + 2\Delta\,(mn\alpha xu' + ln\beta yu'' + lm\gamma zu''') - (mn\alpha^2x + nl\beta^2y + lm\gamma^2z)\,u.$$

If also a_1, b_2, c_3, a_2, a_3, b_1, b_3, c_1, c_2, be written for

$$\frac{1}{p\,(p-1)(p-2)}\frac{d^3u}{dx^3} \cdots\cdots \frac{1}{p\,(p-1)(p-2)}\frac{d^3u}{dy\,dz^2},$$

and a', b', c', f', g', h' for $\frac{1}{p \ldots p-3}\frac{d^4u}{dx^4}$ ..... $\frac{1}{p \ldots p-3}\frac{d^4u}{dy^2\,dz^2}$, then

$$xyzD^3u = -\Delta^2\,\{(m\gamma - n\beta)(mnx\alpha_1 + nly\beta_1 + lmz\gamma_1) + (n\alpha - l\gamma)(mnx\alpha_2 + nly\beta_2 + lmz\gamma_2) + (l\beta - m\alpha)(mnx\alpha_3 + \ldots)\}$$
$$+ 2\Delta\,\{(m\gamma - n\beta)(mn\alpha x\alpha + nl\beta yh + lm\gamma zg) + \ldots\}$$
$$- (mn\alpha^2x + \ldots)\,\{(m\gamma - n\beta)\,u' + \ldots\},$$

$$\text{and } x^2y^2z^2D^4u = \Delta^4\,(m^2n^2x^2\alpha' + \ldots + 2mn\,.\,nlxyh')$$
$$- 4\Delta^3\,\{mnx\alpha\,(mnx\alpha_1 + nly\beta_1 + lmz\gamma_1) + nly\beta\,(mnx\alpha_2 + \ldots) + lmz\gamma\,(mnx\alpha_3 + \ldots)\}$$
$$+ \Delta^2\,\{4\,(m^2n^2\alpha^2x^2\alpha + \ldots + 2mn\,.\,nl\alpha\beta xyh) + 2\,(mn\alpha^2x + nl\beta^2y + lm\gamma^2z)(mnx\alpha + nly\beta + lmz\gamma)\}$$
$$- 4\Delta\,(mn\alpha^2x + nl\beta^2y + lm\gamma^2z)(mn\alpha xv' + nl\beta yv'' + lm\gamma zv''')$$
$$+ (mn\alpha^2x + nl\beta^2y + lm\gamma^2z)^2\,u.$$

III. *Formation of* $Dv \times Dw$.

If w is a second ternary form of order q', then

$$xyz\,qq'Dv \times Dw = xyz\left\{(n\beta - m\gamma)^2\frac{dv}{dx}\frac{dw}{dx} + (l\gamma - n\alpha)^2\frac{dv}{dy}\frac{dw}{dy} + (m\alpha - l\beta)^2\frac{dv}{dz}\frac{dw}{dz}\right.$$
$$+ (l\gamma - n\alpha)(m\alpha - l\beta)\left(\frac{dv}{dy}\frac{dw}{dz} + \frac{dv}{dz}\frac{dw}{dy}\right)$$
$$+ (m\alpha - l\beta)(n\beta - m\gamma)\left(\frac{dv}{dz}\frac{dw}{dx} + \frac{dv}{dx}\frac{dw}{dz}\right)$$
$$\left. + (n\beta - m\gamma)(l\gamma - n\alpha)\left(\frac{dv}{dx}\frac{dw}{dy} + \frac{dv}{dy}\frac{dw}{dx}\right)\right\},$$

in which the coefficient of α^2

$$= n^2xyz\frac{dv}{dy}\frac{dw}{dy}+m^2xyz\frac{dv}{dz}\frac{dw}{dz}-mnxyz\left(\frac{dv}{dy}\frac{dw}{dz}+\frac{dv}{dz}\frac{dw}{dy}\right)$$

$$=-(mnxy^2+nlx^2y)\frac{dv}{dy}\frac{dw}{dy}-(mnxz^2+lmx^2z)\frac{dv}{dz}\frac{dw}{dz}-mnxyz\frac{dv}{dy}\frac{dw}{dz}$$

$$-mnx\left(qv-x\frac{dv}{dx}-y\frac{dv}{dy}\right)\left(q'w-x\frac{dw}{dx}-z\frac{dw}{dz}\right)$$

$$=-\left(mnx\frac{dv}{dx}\frac{dw}{dx}+nly\frac{dv}{dy}\frac{dw}{dy}+lmz\frac{dv}{dz}\frac{dw}{dz}\right)x^2$$

$$+mn\left(qv\frac{dw}{dx}+q'w\frac{dv}{dx}\right)x^2-qq'mnxvw.$$

The coefficient of $\beta\gamma$ in the same

$$=-2mn_x\frac{dv}{dx}\frac{dw}{dx}yz-l^2x\left(\frac{dv}{dy}\frac{dw}{dz}+\frac{dv}{dz}\frac{dw}{dy}\right)$$

$$+lmxyz\left(\frac{dv}{dz}\frac{dw}{dx}+\frac{dv}{dx}\frac{dw}{dz}\right)+nlxyz\left(\frac{dv}{dx}\frac{dw}{dy}+\frac{dv}{dy}\frac{dw}{dx}\right),$$

which becomes, after substituting $lmy+nlz$ for $-l^2x$, identically,

$$=-2mnx\frac{dv}{dx}\frac{dw}{dx}yz+nl\left\{\left(z\frac{dw}{dz}+x\frac{dw}{dx}\right)\frac{dv}{dy}+\left(z\frac{dv}{dz}+x\frac{dv}{dx}\right)\frac{dw}{dy}\right\}yz$$

$$+lm\left\{\left(x\frac{dw}{dx}+y\frac{dw}{dy}\right)\frac{dv}{dz}+\left(x\frac{dv}{dx}+y\frac{dv}{dy}\right)\frac{dw}{dz}\right\}yz$$

$$=\left\{-2\left(mnx\frac{dv}{dx}\frac{dw}{dx}+nly\frac{dv}{dy}\frac{dw}{dy}+lmz\frac{dv}{dz}\frac{dw}{dz}\right)\right.$$

$$\left.+nl\left(qv\frac{dw}{dy}+q'w\frac{dv}{dy}\right)+lm\left(qv\frac{dw}{dz}+q'w\frac{dv}{dz}\right)\right\}yz.$$

Hence it at once appears that, if, as before, $\Delta=\alpha x+\beta y+\gamma z$,

$$xyz\,qq'Dv\,.\,Dw$$

$$=-\Delta^2\left(mnx\frac{dv}{dx}\frac{dw}{dx}+nly\frac{dv}{dy}\frac{dw}{dy}+lmz\frac{dv}{dz}\frac{dw}{dz}\right)$$

$$+\Delta\left\{qv\left(mn\alpha x\frac{dw}{dx}+nl\beta y\frac{dw}{dy}+lm\gamma z\frac{dw}{dz}\right)+q'w\left(mn\alpha x\frac{dv}{dx}+\ldots\ldots\right)\right\}$$

$$-qq'\left(mn\alpha^2x+nl\beta^2y+lm\gamma^2z\right)vw\,;$$

and, in particular, when $w=v$,

$$xyz\,.\,q^2\,(Dv)^2=-\Delta^2\left\{mnx\left(\frac{dv}{dx}\right)^2+nly\left(\frac{dv}{dy}\right)^2+lmz\left(\frac{dv}{dz}\right)^2\right\}$$

$$+2q\Delta\left(mn\alpha x\frac{dv}{dx}+nl\beta y\frac{dv}{dy}+lm\gamma z\frac{dv}{dz}\right)$$

$$-q^2\left(mn\alpha^2x+nl\beta^2y+lm\gamma^2z\right)v^2.$$

Employing, as above, the abbreviations

$$v'=\frac{1}{q}\frac{dv}{dx},\quad v''=\frac{1}{q}\frac{dv}{dy},\ldots\ldots\quad w'''=\frac{1}{q'}\frac{dw}{dz},\quad mnx=\lambda,\ldots\ldots\quad lmz=\nu,$$

$qQ = \lambda\alpha \frac{d}{dx} + \mu\beta \frac{d}{dy} + \nu\gamma \frac{d}{dz}$, so that $Qv = \lambda\alpha v' + \mu\beta v'' + \nu\gamma v'''$,

$R = \lambda\alpha^2 + \mu\beta^2 + \nu\gamma^2$; then

$$xyz\,Dv\,Dw = -\Delta^2(\lambda v'w' + \mu v''w'' + \nu v'''w''') + vQw + wQv - Rvw \ldots\ldots (11),$$

$$xyz\,(Dv)^2 = -\Delta^2(\lambda v'^2 + \mu v''^2 + \nu v'''^2) + 2vQv - Rv^2 \ldots\ldots (12).$$

When it is desired to obtain the product of the results of operating with two odd powers of D, *e.g.*, the product

$$Du \,.\, D^3u,$$

the course to be adopted is obvious; viz.,

$$xyz\,Du \,.\, D^3u = xyz\,Du \,.\, D\,(D^2u),$$

where D^2u is of the order $p-2$ in xyz,

$$= -\Delta^2\left(\lambda u' \frac{1}{p-2}\frac{dD^2u}{dx} + \mu u'' \frac{1}{p-2}\frac{dD^2u}{dy} + \nu u''' \frac{1}{p-2}\frac{dD^2u}{dz}\right)$$
$$+ \Delta\left\{u\left(\frac{\lambda\alpha}{p-2}\frac{dD^2u}{dx} + \ldots\ldots\right) + D^2u\,(\lambda\alpha u' + \ldots\ldots)\right\}$$
$$- Ru\,D^2u;$$

$$x^2y^2z^2\,Du \,.\, D^3u$$
$$= -\Delta^2\,(\lambda u'\,xyz\,D^2u' + \mu u''\,xyz\,D^2u'' + \nu u'''\,xyz\,D^2u'''$$
$$+ \Delta\,\{u\,(\lambda\alpha\,xyz\,D^2u' + \ldots\ldots) + xyz\,D^2u\,(\lambda\alpha u' + \ldots\ldots)$$
$$- Ru\,xyz\,D^2u$$
$$= \Delta^4\,(\lambda u'Pu' + \mu u''Pu'' + \nu u'''Pu''')$$
$$- \Delta^3\,\{2\,(\lambda u'Qu' + \mu u''Qu'' + \nu u'''Qu''')$$
$$- u\,(\lambda\alpha Pu' + \mu\beta Pu'' + \nu\gamma Pu''') + (\lambda\alpha u' + \ldots)\,Pu\}$$
$$+ \Delta^2\,\{R\,(\lambda u'^2 + \mu u''^2 + \nu u'''^2) + 2u\,(\lambda\alpha Qu' + \mu\beta Qu'' + \nu\gamma Qu''')$$
$$+ 2\,(\lambda\alpha u' + \ldots)\,Qu + Ru\,Pu\}$$
$$- \Delta\,\{2uR\,(\lambda\alpha u' + \ldots) + 2Ru\,Qu\} + R^2u.$$

The invariants of (4), when x, y, z are eliminated, are of course contravariants of u; and similarly those of systems of two or more equations of the form (4). It may be of interest to shew how these are formed.

From (9), (12), it at once follows that

$$xyz\,\{uD^2u - (Du)^2\}$$
$$= \Delta^2\,\{mnx\,(u'^2 - au) + nly\,(u''^2 - bu) + lmz\,(u'''^2 - cu)\}$$
$$= \Delta^2\,\{mnx\,(-Cy^2 - Bz^2 + 2Fyz) + nly\,(-Az^2 - Cx^2 + 2Gzx)$$
$$+ lmz\,(-Bx^2 - Ay^2 + 2Hxy)\},$$

where $A = bc - h^2$, ... $H = -ch + fg$; whence, in virtue of

$$-lmy^2z - nlyz^2 = l^2xyz, \quad -lmzx^2 - mnxz^2 = m^2xyz,$$
$$-mnxy^2 - nlyx^2 = n^2xyz,$$

$$uD^2u - (Du)^2 = \Delta^2\,(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm).$$

Again, if $v', v'', v''', a', \ldots\ldots h'$ stand for $\frac{1}{q}\frac{dv}{dx} \ldots\ldots \frac{1}{q(q-1)}\frac{d^2v}{dx\,dy}$,

$$xyz\,\{uD^2v+vD^2u-2Du\,.\,Dv\}$$
$$= -\Delta^2\{mnx\,(av+a'u-2u'v')+nly\,(bv+b'v'-2u''v'')$$
$$+nlz\,(cv+c'u-2u'''v''')\},$$

in which, writing $u'=ax+hy+gz$, $v'=a'x+\ldots$, $u=ax^2+\ldots+2hxy$, $v=a'x^2+\ldots+2h'xy$, and reducing, in virtue of $lx+my+nz=0$, there results

$$uD^2v+vD^2u-2Du\,.\,Dv$$
$$= \Delta^2\,(A_1l^2+B_1m^2+C_1n^2+2F_1mn+2G_1nl+2H_1lm),$$

where $A_1=bc'+b'c-2hh'$, $\ldots\ldots$ $H_1=hg'+h'g-af'-a'f$.

Lastly $xyz\,\{D^2u\,(vDw-wDv)+D^2v\,(wDu-uDw)+D^2w\,(uDv-vDu)\}$

$$= xyz\begin{vmatrix} a & \beta & \gamma & 0 & 0 \\ l & m & n & 0 & 0 \\ u' & u'' & u''' & D^2u & u \\ v' & v'' & v''' & D^2v & v \\ w' & w'' & w''' & D^2w & w \end{vmatrix} = \begin{vmatrix} a & \beta & \gamma & 0 & 0 \\ l & m & n & 0 & 0 \\ u' & u'' & u''' & -\Delta^2Pu+2\Delta Qu-Ru & u \\ v' & v'' & v''' & -\Delta^2Pv+2\Delta Qv-Rv & v \\ w' & w'' & w''' & -\Delta^2Pw+2\Delta Qw-Rw & w \end{vmatrix}$$

Adding to the fourth column the fifth multiplied by R, the first by $mn\Delta^2-2mn\alpha x\Delta$, the second by $nl\Delta^2-2nl\beta y\Delta$, and the third by $lm\Delta^2-2lm\gamma z\Delta$, and subtracting from the fifth column the first multiplied by x, the second by y, and the third by z, the determinant becomes equal to

$$\begin{vmatrix} a & \beta & \gamma & \ldots\ldots & -\Delta \\ l & m & n & -lmn\Delta^2 & 0 \\ u' & u'' & u''' & \Delta^2\,(l^2f\,x+m^2g\,y+n^2h\,z) & 0 \\ v' & v'' & v''' & \Delta^2\,(l^2f'x+m^2g'y+n^2h'z) & 0 \\ w' & w'' & w''' & \Delta^2\,(l^2f''x+m^2g''y+n^2h''z) & 0 \end{vmatrix} = -\Delta^3 \begin{vmatrix} l & m & n & -lmn \\ u' & u'' & u''' & l^2f\,x+\ldots \\ v' & v'' & v''' & l^2f'x+\ldots \\ w' & w' & w''' & l^2f''x+\ldots \end{vmatrix}$$

in virtue of $u'=ax+\ldots\ldots$, $w'''=g''x+f''y+h''x$,

$$lx+my+nz=0\,;$$

whence

$$|\,D^2u,\,Du,\,u\,| = \Delta^3\{\,|\,b,\,c,f\,|\,l^3+\ldots+(\,|\,a,\,b,\,c\,|\,-4\,|\,f,\,g,\,h\,|\,)\,lmn\}.$$

These well-known results afford verifications of the formulæ above investigated.

IV. *Deduction of* D'^2v, $D'v\times D'w$.

When the transversal touches a curve $u=0$, of order p, in the point xyz, so that $l=\frac{1}{p}\frac{du}{dx}=u'$, $m=\frac{1}{p}\frac{du}{dy}=u''$, $n=\frac{1}{p}\frac{du}{dz}=u'''$,

$u=0$; since for all values of xyz

$$u''u'''x = xyzA+x^2(\ xF-yG-zH)+xfu,$$
$$u'''u'y = xyzB+y^2(-xF+yG-zH)+ygu,$$
$$u'u''z = xyzC+z^2(-xF-yG+zH)+zhu,$$

where $$a \ldots h = \frac{1}{p(p-1)}\frac{d^2u}{dx^2} \ldots\ldots \frac{1}{p(p-1)}\frac{d^2u}{dxdy},$$

and $$A \ldots H = bc-f^2 \ldots\ldots fg-ch;$$

if also $$v' = \frac{1}{q}\frac{dv}{dx} \ldots\ldots, \qquad a' \ldots h' = \frac{1}{q(q-1)}\frac{d^2v}{dx^2} \ldots\ldots,$$

in this case—when $u=0$—

$$Pv = mnxa'+nlyb'+lmzc' = xyz(Aa'+Bb'+Cc'+2Ff'+2Gg'+2Hh')$$
$$-(xF+yG+zH)(a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy)$$
$$+2\{x^2F(a'x+h'y+g'z)+y^2G(h'x+b'y+f'z)+z^2H(g'z+f'y+c'z)\},$$

$$Qv = mna v'+nl\beta v''+lm\gamma v'''$$
$$= xyz\{(Aa+H\beta+G\gamma)v'+(Ha+B\beta+F\gamma)v''+(Ga+F\beta+C\gamma)v'''\}$$
$$-\Delta(xF+yG+zH)(v'x+v''y+v'''z)+\Delta(x^2Fv'+y^2Gv''+z^2Hv''')$$
$$+(ax^2F+\beta y^2G+\gamma z^2H)(v'x+v''y+v'''z),$$

$$R = mna^2x+nl\beta^2y+lm\gamma^2z$$
$$= xyz(Aa^2+B\beta^2+C\gamma^2+2F\beta\gamma+2G\gamma a+2Ha\beta)$$
$$-\Delta^2(xF+yG+zH)+2\Delta(ax^2F+\beta y^2G+\gamma z^2H),$$

identically; so that

$$-\Delta^2Pv+2\Delta Qv-Rv$$
$$= xyz[-\Delta^2(Aa'+\ldots+2Hh')$$
$$+2\Delta\{(Aa+\ldots)v'+(Ha+\ldots)v''+(Ga+\ldots)v'''\}-(Aa^2+\ldots)v] \quad \ldots\ldots\ldots (13).$$

In this case also

$$mnxv'w'+nlyv''w''+lmzv'''w'''$$
$$= xyz\{Av'w'+Bv'w''+Cv'''w'''$$
$$+F(v''w'''+v'''w'')+G(v'''w'+v'w''')+H(v'w''+v''w')\}$$
$$+(v'x+v''y+v'''z)(x^2Fw'+y^2Gw''+z^2Hw''')+(w'x+\ldots)(x^2Fv'+\ldots)$$
$$-(xF+yG+zH)vw,$$

$$vQw+wQv = xyz\{(Aa+\ldots)(vw'+wv')+\ldots\}$$
$$-2\Delta(xF+\ldots)vw+\Delta\{x^2F(vw'+wv')+\ldots\}+2(ax^2F+\ldots)vw,$$

and consequently

$$-\Delta^2(mnv'w'+\ldots)+\Delta(vQw+wQv)-Rvw$$
$$= xyz[-\Delta^2\{Av'w'+\ldots+H(v'w''+w'v'')\}$$
$$+\Delta\{(Aa+H\beta+G\gamma)(vw'+wv')+\ldots\}-(Aa^2+\ldots+2Ha\beta)vw] \ldots (14).$$

Hence, distinguishing this special form of D by marking it, if

$$q\,(q-1)\,\ldots\ldots\,(q-r+1)\,D'^r v$$
$$= \left\{(u'\gamma - u''\beta)\frac{d}{dx} + (u''\alpha - u'\gamma)\frac{d}{dy} + (u'\beta - u''\alpha)\frac{d}{dz}\right\}^r v,$$

$u'u''u'''$ being considered as constant in differentiating, and if, q being the order of the subject,

$$q\,(q-1)\,\ldots\ldots\,(q-2r+1)\,P'^r$$
$$= \left(A\frac{d^2}{dx^2} + B\frac{d^2}{dy^2} + C\frac{d^2}{dz^2} + 2F\frac{d^2}{dy\,dz} + 2G\frac{d^2}{dz\,dx} + 2H\frac{d^2}{dx\,dy}\right)^r,$$

$$q\,(q-1)\,\ldots\ldots\,(q-r+1)\,Q'^r$$
$$= \left((A\alpha + H\beta + G\gamma)\frac{d}{dx} + (H\alpha + B\beta + F\gamma)\frac{d}{dy} + (G\alpha + F\beta + C\gamma)\frac{d}{dz}\right)^r,$$

$R' = A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta$;

then (9), (13) $D'^2 v = (-\Delta^2 P' + 2\Delta Q' - R')\,v$ .....(15),

and (11), (14) $D'v\,.\,D'w = -\Delta^2\{Av'w' + \ldots\ldots + H(v'w'' + w'v'')\}$
$+\Delta\,(vQ'w + wQ'v) - R'vw$ .....(16),

$(D'v)^2 = -\Delta^2(Av'^2 + \ldots\ldots + 2Hv'v'') + 2\Delta vQ'v - R'v^2$...(17),

$A \ldots H$ being regarded as constant until after differentiating.

This operator "D'" I have elsewhere ventured to call "Hesse's"; and the results just obtained as particular cases of the more general formulæ, proved in the preceding section of this paper, are well known also now through the skilful analysis by which they have been deduced in Ch. 9 of the second edition of Salmon's "Higher Plane Curves." Those more general formulæ are applicable to questions which cannot be touched by Hesse's more restricted cases; but as I shall have occasion later on to employ the latter, as well as the former, I have thought that it might be to the convenience of the reader to have them collected, with demonstrations, in this paper, and treated as parts of the systematic method, the development and illustration of which it has as its object.

In the special case of $v = u$, if $W = abc + 2fgh - af^2 - bg^2 - ch^2$,

$$D'^2 u = -\Delta^2 W;$$

since, in this case, $A\alpha + Bb + Cc + 2Ff + 2Gg + 2Hh = 3W$,

and $(A\alpha + H\beta + G\gamma)\,u' + (H\alpha + B\beta + F\gamma)\,u'' + (G\alpha + F\beta + C\gamma) = \Delta W$.

To form D'^3: let

$$a_1 = \frac{1}{p\,(p-1)(p-2)}\frac{d^3u}{dx^3}\ldots\ldots, \quad b_3 = \frac{1}{p\,(p-1)(p-2)}\frac{d^3u}{dy^2\,dz}\ldots\ldots,$$
$$e = \frac{1}{p\,(p-1)(p-2)}\frac{d^3u}{dx\,dy\,dz}.$$

$$D'^3u = D'^2(D'u) = -\Delta^2 P'(D'u) + 2\Delta Q'(D'u) - R'D'u$$

$$\begin{aligned}= -\ \Delta^2 \{&(u''\gamma - u'''\beta)(Aa_1 + Bb_1 + Cc_1 + 2Fe + 2Ga_2 + 2Ha_3)\\ &+ (u'''\alpha - u'\gamma)(Aa_2 + Bb_2 + Cc_2 + 2Fb_3 + 2Ge + 2Hb_1)\\ &+ (u'\beta - u''\alpha)(Aa_3 + Bb_3 + Cc_3 + 2Fc_2 + 2Gc_1 + 2He)\}\\ +2\Delta\ \{&(Aa + H\beta + G\gamma)[(u''\gamma - u'''\beta)\,a + (u'''\alpha - u'\gamma)\,h + (u'\beta - u''\alpha)\,g]\\ &+ (Ha + B\beta + F\gamma)[(u''\gamma - u'''\beta)\,h + (u'''\alpha - u'\gamma)\,b + (u'\beta - u''\alpha)\,f]\\ &+ (Ga + F\beta + C\gamma)[(u''\gamma - u'''\beta)\,g + (u'''\alpha - u'\gamma)\,f + (u'\beta - u''\alpha)\,c]\}\\ -(Aa^2 + \ldots\ldots + 2Ha\beta)&\{(u''\gamma - u'''\beta)\,u' + \ldots\ldots\};\end{aligned}$$

which vanishes identically.

If $W' = \dfrac{1}{3(p-2)}\dfrac{dW}{dx}$, $W'' = \dfrac{1}{3(p-2)}\dfrac{dW}{dy}$, $W''' = \dfrac{1}{3(p-2)}\dfrac{dW}{dz}$,

the term multiplied by $-\Delta^2$ is equal to

$$3\{(u''\gamma - u'''\beta)\,W' + (u'''\alpha - u'\beta)\,W'' + (u'\beta - u''\alpha)\,W'''\};$$

and if, in the term multiplied by 2Δ, $bc - f^2 \ldots fg - ch$ are substituted for $A \ldots H$, it reduces identically to

$$W\{\alpha(u''\gamma - u'''\beta) + \beta(u'''\alpha - u'\gamma) + \gamma(u'\beta - u''\alpha)\},$$

which vanishes identically; hence

$$D'^3u = 3\Delta^2\{\alpha(u''W''' - u'''W'') + \beta(u'''W' - u'W''') + \gamma(u'W'' - u''W')\} \quad \ldots\ldots\ldots(18).$$

To form D'^4u: since xyz make $u = 0$,

$$\begin{aligned}D'^4u &= (-\Delta^2 P' + 2\Delta Q' - R')^2\,u\\ &= \Delta^4 P'^2 u - 4\Delta^3 P'Q'u + \Delta^2(2R'P'u + 4Q'^2u) - 4\Delta R'Q'u\\ &= \Delta^4 P'^2 u - 4\Delta^3 P'Q'u + 6\Delta^2 R'W,\end{aligned}$$

since $Q'^2u = R'W$, $Q'u = \Delta W$, $P'u = 3W$.

Now $$P'^2u = p \ldots\ldots (p-3)\left(A\frac{d^2}{dx^2} + \ldots\ldots + 2H\frac{d^2}{dx\,dy}\right)^3 u,$$

or $$P'^2u = A^2a' + \ldots\ldots + (4F^2 + 2BC)f' + \ldots\ldots + (4AF + 8GH)i' + \ldots\ldots + 4AHa_2' + \ldots\ldots + 4CFc_2',$$

if $$a' = \frac{1}{p \ldots (p-3)}\frac{d^4u}{dx^4}, \quad \ldots\ldots \quad f' = \frac{1}{p \ldots (p-3)}\frac{d^4u}{dy^2\,dz^2}, \quad \ldots\ldots$$

$$i' = \frac{1}{p \ldots (p-3)}\frac{d^4u}{dx^2\,dy\,dz}, \quad \ldots\ldots\ a_2' = \frac{1}{p \ldots (p-3)}\frac{d^4u}{dx^3\,dy} \ldots\ldots$$

Again $$p \ldots (p-2)\,Q'P'u$$

$$= \left\{(Aa + \ldots)\frac{d}{dx} + (Ha + \ldots)\frac{d}{dy} + (Ga + \ldots)\frac{d}{dz}\right\}\left(A\frac{d^2u}{dx^2} + \ldots\right);$$

whence $Q'P'u = 3\{(Aa + \ldots)\,W' + (Ha + \ldots)\,W'' + (Ga + \ldots)\,W'''\}$.

Thus $D'^4u = \Delta^4 (A^2a' + \ldots + 4CFc'_2) - 12\Delta^3 \{(Aa + \ldots) W' + \ldots\}$
$+ 6\Delta^2 (Aa^2 + \ldots + 2Ha\beta) W^* \ldots\ldots (19).$

I select, as applications of the method just developed, the discussion of two questions on the general quartic curve hitherto unapproached, as far as I am aware; the first being that of

V. *The Inflexion-Tangential Points,*

i.e., the points in which the tangents at points of inflexion again meet the quartic. If (xyz) is a point of inflexion, $(x'y'z')$ the point in which the tangent at (xyz) again meets u, then (6)

$$4 (u''\gamma - u'''\beta) D'^3 + (x' - x) D'^4 u = 0,$$

or
$$x' D'^4 u = x D'^4 u - 4 (u''\gamma - u'''\beta) D'^3 u.$$

Now (19) $D'^4 u = \Delta^4 (A^2a' + \ldots) - 12\Delta^3 \{(Aa + \ldots) W' + \ldots\}$,

since $W = 0$, and (18)

$$D'^3 u = 3\Delta^2 \{a (u'' W''' - u''' W'') + \beta (u''' W' - u' W''') + \gamma (u' W'' - u'' W')\}.$$

But for all values of xyz, remembering that $\Delta = \alpha x + \beta y + \gamma z$,

$$x \{a (u'' W''' - u''' W'') + \ldots\}$$
$$= \Delta (u'' W''' - u''' W'') + u (W''\gamma - W'''\beta) - W (u''\gamma - u'''\beta);$$

hence, in this case, $xD'^3 u = 3\Delta^3 (u'' W''' - u''' W'')$;

and $xx' D'^4 u = x\Delta^4 (A^2a' + \ldots)$
$$- 12\Delta^3 [x \{(Aa + \ldots) W' + \ldots\} + (u''\gamma - u'''\beta)(u'' W''' - u''' W'')].$$

Substituting, in the term multiplied by Δ^3,

$yW'' + zW'''$ for $-xW'$, and $hx + by + fz$ for u'', $gx + fy + cz$ for u''', it becomes identically equal to

$$\Delta \{(Ay - Hx) W'' + (Az - Gx) W'''\}$$
$$+ \{(f\gamma - c\beta) W'' + (f\beta - b\gamma) W'''\} (ax^2 + \ldots\ldots + 2hxy),$$

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* There is an apparent discrepancy between this and the developments of $D'^4 u$ previously obtained in the term multipled by Δ^4. In a short communication to the London Mathematical Society (Proc. viii. 219), on "Hesse's Ternary Operator," I pointed out an equality of two forms, for values of xyz making $u = 0$, which reconciles this apparent discrepancy. I have since verified that, for all values of xyz,

$$\left(A \frac{d^2}{dx^2} + \ldots\ldots + 2H \frac{d^2}{dx\,dy}\right)^2 u = A \frac{d^2 W}{dx^2} + \ldots\ldots + 2H \frac{d^2 W}{dx\,dy} - p (p-1)(p-2)\, uS,$$

where S is the quartic invariant of the polar cubic of u, A being

$$\frac{d^2u}{dy^2} \frac{d^2u}{dz^2} - \left(\frac{d^2u}{dy\,dz}\right)^2 \ldots\ldots, \quad \text{and } W, \ \frac{d^2u}{dx^2} \frac{d^2u}{dy^2} \frac{d^2u}{dz^2} + \ldots\ldots - \frac{d^2u}{dz^2} \left(\frac{d^2u}{dx\,dy}\right)^2.$$

which, when $W=0$, $u=0$, is equal to

$$-\Delta x\,(AW'+HW''+GW'''),$$

and, finally,

$$x'D'^4u=\Delta^4\{x\,(A^2a'+\ldots\ldots+4CFc_2')-12\,(AW'+HW''+GW''')\},$$

with analogous values for $y'D'^4u$, $z'D'^4u$, so that

$$\begin{aligned}x' : y' : z' = {} & x\,(A^2a'+\ldots\ldots)-12\,(AW'+HW''+GW''')\\ & : y\,(A^2a'+\ldots\ldots)-12\,(HW'+BW''+FW''')\\ & : z\,(A^2a'+\ldots\ldots)-12\,(GW'+FW''+CW''')\ldots(20),\end{aligned}$$

where

$$\begin{aligned}u=a'x^4+b'y^4+c'z^4+{}& 6f'y^2z^2+6g'z^2x^2+6h'x^2y^2\\ &+12i'x^2yz+12j'y^2zx+12k'z^2xy\\ &+4a_2'x^3y+4a_3'x^3z+4b_1'y^3x+4b_3'y^3z+4c_1'z^3x+4c_2'z^3y,\end{aligned}$$

$$a\ldots h=\tfrac{1}{12}\frac{d^2u}{dx^2}\ldots\tfrac{1}{12}\frac{d^2u}{dx\,dy},\quad A\ldots H=bc-f^2\ldots fg-ch;$$

and

$$W=abc+2fgh-af^2-bg^2-ch^2.$$

The determination of the Inflexion-Tangential Curve would be effected, if practicable, by the elimination of xyz among the two equations (20) combined with $u=0$. But as, in the simplest examples, this would be an impracticably laborious process, I have attempted the determination of the equation by regarding xyz as a point on the curve from which $lx+my+nz$ can be drawn so as to meet u in a point of inflexion.

This supposition leads to two relations among l, m, n and the coordinates of the former point, which are independent of α, β, γ. The elimination of l, m, n from these, combined with

$$xl+ym+zn=0,$$

would therefore evidently give the required equation of the Inflexion-Tangential curve.

If $lx+my+nz$ is a tangent at a point of inflexion on the quartic $u=0$, meeting the curve again in the point xyz, then (4)

$$4Du+6\rho D^2u+4\rho^2D^3u+\rho^3D^4u=0$$

must have three equal roots; the conditions for which are—say,

$$3\,(D^2u)^2-4Du\,.\,D^3u=0,$$

and

$$(D^2u)^3-2\,(Du)^2\,D^4u=0.$$

To save the inconvenience of referring back, I reproduce here the formula (11),

$$xyz\,Dv\,.\,Dw=-\Delta^2\,(\lambda v'w'+\mu v''w''+\nu v'''w''')+\Delta\,(vQw+wQv)-Rvw,$$

where, q, q' being the orders of v and w,

$$v' = \frac{1}{q}\frac{dv}{dx}, \quad \ldots\ldots \quad w''' = \frac{1}{q'}\frac{dw}{dz};$$

also $\lambda = mnx$, $\mu = nly$, $\nu = lmz$, and Qw stands for

$$\frac{1}{q'}\left(\alpha\lambda\frac{d}{dx} + \beta\mu\frac{d}{dy} + \gamma\nu\frac{d}{dz}\right)w \ldots\ldots, \quad R = \lambda\alpha^2 + \mu\beta^2 + \nu\gamma^2.$$

Hence
$$xyz\,Du\,.\,D^3u = -\Delta^2(\lambda u'D^2u' + \mu u''D^2u'' + \nu u'''D^2u''') + \Delta\,(uD^2Qu + D^2u\,.\,Qu) - RuD^2u,$$

the terms in which having u as a factor vanish since $u = 0$.

Again,
$$xyz\,D^3v = -\Delta^2 Pv + 2\Delta Qv - Rv,$$

where
$$Pv = \frac{1}{q\,(q-1)}\left(\lambda\frac{d^2}{dx^2} + \mu\frac{d^2}{dy^2} + \nu\frac{d^2}{dz^2}\right)v;$$

so that
$$x^2y^2z^2\,.\,Du\,.\,D^3u = \Delta^4(\lambda u'Pu' + \mu u''Pu'' + \nu u'''Pu''') - \Delta^3(2\lambda u'Qu' + 2\mu u''Qu'' + 2\nu u'''Qu''' + Pu\,.\,Qu) + \Delta^2\{\lambda u'Ru' + \mu u''Ru'' + \nu u'''Ru''' + 2\,(Qu)^2\}.$$

Hence $3\,(D^2u)^2 - 4DuD^3u = 0$ gives

$$\Delta^2\{3\,(Pu)^2 - 4\,(\lambda u'Pu' + \mu u''Pu'' + \nu u'''Pu''')\} + 8\Delta\,(\lambda u'Qu' + \mu u''Qu'' + \nu u'''Qu''' - Pu\,.\,Qu) + 4\{(Qu)^2 - R\,(\lambda u'^2 + \mu u''^2 + \nu u'''^2)\} = 0 \ldots\ldots (21).$$

Now
$$2\Delta\,(\lambda u'Qu' + \ldots\ldots - Pu\,.\,Qu) + (Qu)^2 - R\,(\lambda u'^2 + \ldots\ldots)$$

is easily transformed into

$$\Delta^2\{\mu\nu\,(f^2 - bc) + \nu\lambda\,(g^2 - ca) + \lambda\mu\,(h^2 - ab)\},$$

in virtue of the relations

$$lx + my + nz = 0, \quad u = 0, \quad u' = ax + hy + gz, \quad u'' = hx + \ldots, \quad u''' = gx + \ldots,$$

so that
$$3\,(Pu)^2 - 4\{\lambda u'Pu' + \mu u''Pu'' + \nu u'''Pu''' + \mu\nu\,(bc - f^2) + \nu\lambda\,(ca - g^2) + \lambda\mu\,(ab - h^2)\} = 0.$$

Finally, since $Pu = \lambda a + \mu b + \nu c$, $Pu' = \lambda a_1 + \mu b_1 + \nu c_1$, $Pu'' = \lambda a_2 + \ldots$, $Pu''' = \lambda a_3 + \ldots$, where $a_1 = \frac{1}{2}\frac{1}{4}\frac{d^3u}{dx^2}\ldots$, $c_2 = \frac{1}{2}\frac{1}{4}\frac{d^3u}{dz^2\,dy}$, the first condition of the equation in ρ having three equal roots is

$$(3a^2 - 4u'a_1)\lambda^2 + (3b^2 - 4u''b_2)\mu^2 + (3c^2 - 4u'''c_3)\nu^2 + 2\,(bc + 2f^2 - 2u''c_2 - 2u'''b_3)\mu\nu + 2\,(ca + 2g^2 - 2u'''a_3 - 2u'c_1)\nu\lambda + 2\,(ab + 2h^2 - 2u'b_1 - 2u''a_2)\lambda\mu = 0 \ldots\ldots (22).$$

The second condition, viz., $(D^2u)^3 - 2\,(Du)^2\,D^4u = 0$,

gives, after division by Δ^3, since

$$D^2u = -\Delta^2 Pu + 2\Delta Qu, \quad (Du)^2 = -\Delta^2(\lambda u'^2 + \mu u''^2 + \nu u'''^2),$$

when $u=0$, $-\Delta^3(Pu)^3+6\Delta^2(Pu)^2Qu-12\Delta Pu(Qu)^2+8(Qu)^3$
$+2(\lambda u'^2+\mu u''^2+\nu u'''^2)\{\Delta^3P^2u-4\Delta^2PQu+\Delta(4Q^2u+2RPu)-4RQu\}=0$,

subtracting from this the first condition, in the form (21),

$$\Delta^2\{(Pu)^2-4(\lambda u'Pu'+\ldots\ldots)\}+8\Delta(\lambda u'Qu'+\ldots\ldots-Pu\,.\,Qu)$$
$$+4\{(Qu)^2-(\lambda u'^2+\ldots\ldots)R\}=0,$$

multiplied by $2Q$, after division by Δ,

$$\Delta^2\{2(\lambda u'^2+\ldots)P^2u-(Pu)^3\}+8\Delta\{(\lambda u'Pu'+\ldots)Qu-(\lambda u'^2+\ldots)PQu\}$$
$$+4\{Pu(Qu)^2-4(\lambda u'Qu'+\ldots)Qu+(\lambda u'^2+\ldots)(2Q^2u+RPu)\}=0.$$

The terms $8\Delta\{\ldots\ldots\}+4\{Pu(Qu)^2\ldots\ldots\}$ are together equal to

$$\Delta^2 4lmn\{-4lmnu'u''u'''+(mnxa+nlyb+lmzc)(lu''u'''+mu'''u'+nu'u'')$$
$$+2lyz(mc_2-nb_3)(nu''-mu''')u'+2mzx(na_3-lc_1)(lu'''-nu')u''$$
$$+2nxy(lb_1-ma_2)(mu'-lu'')u'''\};$$

and $P^2u=\lambda^2a'+\mu^2b'+\nu^2c'+2\mu\nu f'+2\nu\lambda g'+2\lambda\mu h'$, $Pu=\lambda a+\mu b+\nu c$,

where $a'=\frac{1}{24}\frac{d^4u}{dx^4}\ldots$, $h'=\frac{1}{24}\frac{d^4u}{dy^2dz^2}$. Making these substitutions, and dividing by Δ^2, the equation above becomes therefore

$$2(mnxu'^2+\ldots)(m^2n^2x^2a'+n^2l^2y^2b'+l^2m^2z^2c'+2nl\,.\,lmyzf'+2lm\,.\,mnzxg'$$
$$+2mn\,.\,nlxyh')-(mnxa+nlyb+lmzc)^3$$
$$+4lmn\{-4lmnu'u''u'''+(mnxa+\ldots)(lu'u''+mu'''u'+nu'u'')$$
$$+2lyz(mc_2-nb_3)(nu''-mu''')+2mzx(na_3-lc_1)(lu'''-nu')$$
$$+2nxy(lb_1-ma_2)(mu'-lu'')\}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23).$$

Substituting in (22) for l from

$$lx+my+nz=0,$$

a binary quartic is formed, reducing to

$$4(u'''m-u''n)(c_3m^3-3c_2m^2n+3b_3mn^2-b_2n_3)=3(cm^2-2fmn+bn^2)^2\ldots(24),$$

after some simple transformations of the coefficients, which render it divisible by y^2z^2. Similarly (23) leads to a binary sextic in mn, which after subtraction of the quartic above, multiplied by $mnxy^2z^2u'$, is divisible by y^3z^3, giving

$$2(u'''m-u''n)^2(c'm^4-4c_2'm^3n+6f'm^2n^2-4b_3'mn^3+b'n^4)$$
$$=(cm^2-2fmn+bx^2)^3\ldots\ldots\ldots(25).$$

Now these two equations are the conditions, when $u=0$, that

$$u+4\rho Ku+6\rho^2K^2u+4\rho^3K^3u+\rho^4K^4u$$

shall be a perfect cube, where K stands for a binary operator; viz.,

$$p(p-1)\ldots\ldots(p-r+1)K=\left(n\frac{d}{dy}-m\frac{d}{dz}\right),$$

p being the order of the subject, and where ρ is any parameter.

Instead of the sextic (25) may be used

$$9\,(cm^2-2fmn+bn^2)(c'm^4-4c'_2m^3n+6f'm^2n^2-4b'_3mn^3+bn^4)$$
$$=8\,(c_3m^3-3c_2m^2n+3b_3mn^2-b_2n^3)^2 \ldots (26),$$

derivable from (23), (24), the coefficients of which are of the second order only in xyz, whereas those of (25) are of the sixth order in xyz.

I have applied the method just explained to the determination of the equation of the Inflexion-Tangential curve in the case of

$$u = ax^4+by^4+cz^4+4ex^3y+4fx^3z,$$

and find it under the form

$$\begin{aligned}&[144a^2bce^2f^2x^5+256x^3\,(bf^3y-ce^3z)^2+72a^2bcefx^2\,(bfy^3+cez^3)\\&\quad+40a^2bcx^2yz\,(bf^3y+ce^3z)+abc\,(9a^2x^3+32efxyz)(bf^2y^2+ce^2z^2)\\&\quad+abc\,\{a^3bc-16\,(bf^4+ce^4)\}\,xy^2z^2-128bce^2f^2\,(bfy^3+cez^3)\,yz\\&\quad-a^2bc\,\{36x^2\,(bf^2y^2+ce^2z^2)+2abcy^2z^2\}\,(ey+fz)\\&\quad+64\,(ce^4-bf^4)(ey-fz)\,y^2z^2]\,(ax+4ey+4fz)=0.\end{aligned}$$

It is a presumption of the correctness of this result that, after casting out the extraneous factors during the calculation, it has $ax+4ey+4fz$ as a factor; as it is easily verified that four points of inflexion, of which two are real when b and c are of opposite signs, lie on the line $x=0$; and that the four tangents at these points meet the curve again on the line

$$ax+4ey+4fz=0.$$

Multiplying out, and dropping the terms

$$\{144a^2bce^2f^2x^2+256\,(bf^3y-ce^3z)^2\}\,(ax^4+by^4+cz^4+4ex^3y+4fx^3z),$$

the equation of the Inflexion-Tangential curve takes the simpler form

$$\begin{aligned}&(9a^4bcx^4-192a^2bcefx^3yz)(bf^2y^2+ce^2z^2)\\&\quad+32a^3bcx^3yz\,(bf^3y+ce^3z)-a^5b^2c^2x^2y^2z^2-2a^4b^2c^2xy^2z^2\,(ey+fz)\\&\quad+8a^3b^2c^2y^2z^2\,(ey+fz)^2+256\,(bf^2y^2+ce^2z^2)^3=0.\end{aligned}$$

It is not probable that the equation of the Inflexion-Tangential for the case of the general quartic can be exhibited without an extraneous factor in terms of differential coefficients of lower order than the fourth; that is, otherwise than in terms of the coefficients themselves.

I have also formed the equation of the Inflexion-Tangential curve in the case of the general quartic transformed so as to be deprived of the terms

$$y^4,\ z^4,\ y^3x,\ y^3z,\ z^3x,\ z^3y,$$

in which case $\dfrac{d^3u}{dy^3}$ or b_2, and $\dfrac{d^3u}{dz^3}$ or c_3, vanish as well as $b'c'b'_3c'_2$.

Equations (24), (26) then give

$$4\,(u'''m-u''n)\,mn\,(c_2m-b_3n)+(cm^2-2fmn+bn^2)^2=0 \ \ldots\ (27),$$

$$3f'\,(cm^2-2fmn+bn^2)-4\,(c_2m-b_3n)^2=0,$$

or

$$(4c_2-3f'c)\,m^2+2\,(3ff'-4b_3c_2)\,mn+(4b_3^2-3f'b)\,n^2=0;$$

while (27), (28) lead to

$$9f'^2mn\,(u'''m-u''n)+4\,(c_2m-b_3n)^3=0,$$

or $4c_2^3\,m^3-3\,(4b_3c_2^2-3f'^2u''')\,m^2n+3\,(4b_3^2c_2-3f'^2u'')\,mn^2-4b_3^3\,n^3=0$...(29).

The resultant of (28), (29), after division by f'^3, is

$$\begin{aligned}27f'\,(3f'b-4b_3^2)(3f'c-4c_2^2)&\\ \times\{(3f'c-4c_2^2)\,u''^2-2\,(3f'f-4b_3c_2)\,u''u'''+(3f'b-4b_3^2)\,u'''^2\}&\\ +6f'\,\{(3f'c-4c_2^2)(c_2^2b-b_3^2c)\,(c_2b-b_3f)-2c_2\,(3f'b-4b_3^2)(b_3c-c_2f)^2\}\,u''&\\ +6f'\,\{(3f'b-4b_3^2)(b_3^2c-c_2^2b)(b_3c-c_2f)-2b_3(3f'c-4c_2^2)(c_2b-b_3f)^2\}\,u'''&\\ +64\,(c_2^2b+b_3^2c-2b_2c_3f)^3=0\ldots\ldots\ldots

equations containing the squares of the modules, and to the case in which the order of the transformation is uneven.

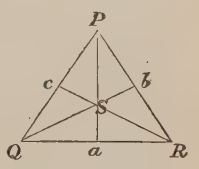
Art. 2.—We represent by q, the square of the modulus of a given elliptic function; by p, the square of the transformed modulus, the transformation being primary,* and of the uneven order N; by

$$(1) \ldots\ldots F(p, q, 1) = 0$$

the modular equation subsisting between p and q; in connection with this equation, we consider the modular curve C, of which the trilinear equation

$$(2) \ldots\ldots F(\alpha, \beta, \gamma) = 0$$

is obtained by writing $p = \frac{\alpha}{\gamma}$, $q = \frac{\beta}{\gamma}$. We denote by P, Q, R the vertices of the triangle $\alpha\beta\gamma$; and by S the point $\alpha = \beta = \gamma$, or $p = q = 1$; the three intersections (PS, QR), (QS, RP), (RS, PQ) we represent by a, b, c. We regard p and q as the



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and the discussion of this equation has recently been resumed by Professor Klein (*Mathematische Annalen*, Vol. xiv., p. 112, May 1878). These geometers have expressed $T(\omega)$ and $T(\Omega)$ rationally in terms of a third indeterminate, in the cases $N = 2, 4$; $3, 5, 7, 13$; the deficiency of the equation (7) being zero in these six cases; but neither of them has given any example of the equation in its explicit form. Writing x for $T(\omega)$ and y for $T(\Omega)$, I find, when $N = 3$,

$$x(x + 2^7 . 3 . 5^3)^3 + y(y + 2^7 . 3 . 5^3)^3 - 2^{16}x^3y^3 + 2^{11} . 3^2 . 31x^2y^2(x + y) - 2^2 . 3^3 . 9907xy(x^2 + y^2) + 2 . 3^4 . 13 . 193 . 6367x^2y^2 + 2^3 . 3^5 . 5^3 . 4471xy(x + y) - 2^{15} . 5^6 . 22973xy = 0.$$

The equations (2) ... (7) are symmetrical with regard to the two indeterminates, and, the number N being uneven, they are of the order $A + B$ (Art. 3) in the indeterminates separately, and of the order $2A$ in the indeterminates jointly. I have recently found that the Eulerian functions $\chi(\omega)$ and $\chi(\Omega)$, defined by the equation

$$\chi(\omega) = \sqrt[6]{2}\, \epsilon^{\frac{i\pi\omega}{24}} \times \prod_1^\infty (1 - (-1)^s \epsilon^{si\pi\omega}),$$

satisfy an equation (1) having the same properties; for the cube $\chi(\omega)$ this had already been shown by M. Kœnigsberger (*Borchardt's Journal*, Vol. lxxii., p. 182 sqq.) Writing $u_1 = \chi(\omega)$, $v_1 = \chi(\Omega)$, we have, in the cases $N = 5, 7, 11$,

$$u_1^6 + 2^{\frac{4}{3}} u_1^5 v_1^5 - 2^{\frac{5}{3}} u_1 v_1 + v^6 = 0,$$

$$u_1^{12} + 8 . 2^{\frac{1}{3}} u_1^{11} v_1^{11} - 44 u_1^9 v_1^9 + 44 . 2^{\frac{2}{3}} u_1^7 v_1^7 - 44 . 2^{\frac{1}{3}} u_1^5 v_1^5 + 22 u_1^3 v_1^3 - 2 . 2^{\frac{2}{3}} u_1 v_1 + v_1^{12} = 0,$$

$$u_1^8 - 16 u_1^7 v_1^7 + 7 u_1^4 v_1^4 - 2 u_1 v_1 + v_1^8 = 0.$$

The function $\chi(\omega)$ is a twenty-fourth root of $u^8(1 - u^8)$; the formulæ relating to its linear transformation have been given by M. Hermite (*Comptes Rendus*, 1858, Vol. xlvi., p. 721). In respect of simplicity of form, the equations (1), (2) ... (7) appear to arrange themselves in this numerical order; but, in respect of simplicity of algebraical theory, the order is reversed, as the deficiency decreases from (1) to (7).—H. J. S. S.]

* A transformation $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ of the uneven order N is primary when it satisfies the congruence $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \equiv \begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$, mod. 2. For the theory of the elliptic multiplier it is convenient to fix the signs of $abcd$ by the additional condition $a \equiv 1$, mod. 4; but for our present purpose this restriction is unnecessary.

parameters of two pencils of lines $\alpha - p\gamma$, $\beta - q\gamma$, of which the centres are Q and P, and between the rays of which a correspondence is established by the equation (1); we observe that, to the values 0, 1, ∞ of either parameter, there answer in the two pencils respectively the rays QR, QS, QP; PR, PS, PQ; and that, by a known property of the modular equation, the pairs of rays (QR, PR), (QS, PS), (QP, PQ) are corresponding rays in the two pencils, either ray of any of these pairs being the only ray answering to the other ray of the same pair; the modular curve C is the locus of the intersections of corresponding rays in the two pencils.*

We denote by m and n the order and class of C; by H its deficiency; by K and I its cuspidal and inflexional indices; by D and T its discriminantal order and class; by $E(p) = E(q)$ the highest, by $E'(p) = E'(q)$ the lowest, exponents of p and q which present themselves in the equation (1).

ARTS. 3—8. *N not divisible by a square.*

Art. 3.—We confine ourselves, in the first instance, to the case in which N is not divisible by any square; and we represent, in this case, by A the sum of the divisors of N which surpass $\sqrt{N}$; by B the sum of the divisors of N which are less than $\sqrt{N}$; by ν the number of divisors of either sort, so that 2ν is the whole number of divisors of N. We then have the formulæ

$$\begin{aligned}
&\text{(i.) .....} && m &&= 2A,\\
&\text{(ii.) .....} && n &&= 3A - B,\\
&\text{(iii.) .....} && H &&= \tfrac{1}{2}(A+B) - 3\nu + 1,\\
&\text{(iv.) .....} && K &&= 2A + 2B - 6\nu,\\
&\text{(v.) .....} && I - K &&= 3(A-B),\\
&\text{(vi.) .....} && D &&= 4A^2 - 5A + B,\\
&\text{(vii.) .....} && T - D &&= (A-B)(5A-B).
\end{aligned}$$

To these we may add the equations

$$\text{(viii.) .....} \begin{cases} E(p) = E(q) = A + B, \\ E'(p) = E'(q) = 2B. \end{cases}$$

Of these formulæ (i.) and (viii.) are well known; of the remainder, it will suffice to attend to (iii.) and (iv.), because, when the values of m, H, and K are given, the values of n, I, D, and T are known from the equations of Plücker.

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* [In a paper which I hope shortly to lay before the Society, I have discussed, with some fulness of detail, the relation of the algebraical singularities of the parametric equation $F(p, q, 1) = 0$ to the characteristic singularities of the curves of which the equations are included in the formula $F\left(\frac{A}{B}, \frac{C}{D}, 1\right) = 0$, A, B, C, D being the equations of straight lines. The discussion comprises an examination of the effect of any quadric transformation on the singularities of a curve.—H. J. S. S.]

Art. 4.—*The Deficiency.*—The demonstration of the formulæ (iii.) and (iv.) depends on the simultaneous expression of the modular parameters p and q as transcendental functions of the quotient of the periods of the given elliptic function. As we have already developed these considerations elsewhere,* we shall in this place assume the results of the discussion as known, and shall confine ourselves to their application to the formulæ (iii.) and (iv.)

Denoting by x and y real quantities, of which y is essentially posi-

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* In an Introduction (now in the press) to Mr. J. W. L. Glaisher's Tables of the Theta Functions. The method indicated in this article has been already employed by Professor Klein, in the paper to which reference has already been made (*Math. Ann.*, Vol. xiv., p. 111; see especially §§ 6—8, 7—13), and by Professor Dedekind, in a letter addressed to M. Borchardt (June 1877, see *Borchardt's Journal*, Vol. lxxxiii., p. 265, especially §§ 1—4 and 7). In the year 1873, I submitted to the Academy of Sciences in Paris a Mémoire *Sur les Équations Modulaires* (*Comptes Rendus*, August 1873, Vol. lxxvii., p. 472). In this Mémoire (which was ultimately presented, without alteration, to the Accademia dei Lincei, and was printed in their *Atti*, Vol. i., series 3, p. 136, February 1877), I had employed the same method (see Arts. ii., ix., and x. of the Mémoire) to establish the relation which exists between the modular equations of order N and the binary quadratic forms of the positive determinant N. The Mémoire was devoted to that theory alone, as I attached more importance to it than to any other result relating to the modular equations at which I had then arrived. But I had already in the year 1873 obtained:—(i.) a proof of the existence of the modular equations, simpler perhaps than that of M. Dedekind, and based solely on those elementary properties of the function $\phi(\omega)$, which were deduced from the theorem of Fourier by Cauchy and Poisson, without employing any elliptic formulæ; (ii.) a determination in the simpler cases of the Plückerian characteristics of the modular curves; (iii.) a solution of one part at least of the problem relating to complex modules, proposed by Jacobi in Art. 32 of the *Fundamenta Nova*. I communicated to Professor Cayley, in 1873, the formulæ for the deficiency of the equations (2) ... (5) when N is an uneven prime (see his Memoir on the Transformation of Elliptic Functions, presented to the Royal Society in that year); the formulæ for the cuspidal index I obtained by transforming into normal developments the parametric developments which give the deficiency (see Art. 5 of this paper); thus, the order of the curves being known, all their Plückerian characteristics were determined. The case when N is a product of uneven primes presents no greater difficulty than the case when N is a prime; and I had (in fact) obtained the formula for this more general case as early as 1873. The case when N is divisible by a square, and still more the case when N is itself a square, appeared to involve some difficulty; and these I left untouched till the spring of the present year, when I found that the introduction of the arithmetical function f' (see Art. 9 of this paper) caused the supposed difficulty to disappear. To the more exact determination of the indices characteristic of each special singularity of the modular curves, I was guided by the methods employed in a former paper on the Higher Singularities of Plane Curves.

A complete system of formulæ, analogous to that given in the present paper for the modular equation (5), I have already obtained for the equations (2), (3), and (4); with the equation (7), and with the eight equations between corresponding powers of $\chi(\omega)$ and $\chi(\Omega)$, I have not advanced equally far, but I have not found that they offer any peculiar difficulty.

In the Mémoire sur les Équations Modulaires, I have confined myself (as in the present paper) to the equation (5) between the squares of the modules. At the time when the Mémoire was written, I was well acquainted with the characteristic property of the function $T'(\omega)$; viz., that it is unchanged by any linear transformation of the elliptic functions; and I even thought of employing it in the Mémoire instead of the function $\phi^8(\omega)$. I had conjectured (erroneously however) that the modular curves T' derived from the equation (7) would represent ordinary periodic continued fractions with positive integral quotients, in the same way in which the

tive, and by ω the complex variable $x+iy$, which is thus subject to the restriction that the coefficient of i in its imaginary part is positive, we define the function $\phi(\omega)$ by the equation

$$(3) \ldots\ldots \phi(\omega) = \sqrt{2} \cdot \epsilon^{\frac{i\pi\omega}{8}} \prod_{s=1}^{s=\infty} \frac{1+\epsilon^{2si\pi\omega}}{1+\epsilon^{(2s-1)i\pi\omega}};$$

and we consider the $A+B$ quantities Ω, of which the values are given by the equation

$$(4) \ldots\ldots \Omega = \frac{q'\omega+2k}{g},$$

g, g' being any two conjugate divisors of N, and k being any term of a complete system of residues for the modulus g. We then have the fundamental theorem,

"If $q = \phi^8(\omega)$, the $A+B$ corresponding values of p are included in the formula $p = \phi^8(\Omega)$;

or, which is the same thing,

$$F(p,\ q,\ 1) = \Pi[p-\phi^8(\Omega)],$$

the sign of multiplication extending to the $A+B$ values of Ω."

It results from the discussion to which we have referred, that, if we regard q as an independent complex variable, and represent its values in the usual manner by the points of a plane, p, considered as a function q, has no spiral points other than the three $q=0$, $q=1$, $q=\infty$ (it will be remembered that in the plane of double algebra the infinitely

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modular curves C derived from the equation (5) represent periodic continued fractions with even quotients. But I was deterred from employing the equation (7) by the consideration that there was not a single calculated example of it; indeed, at that time I was not acquainted with the researches of Dr. Felix Müller, and did not know that the equation had attracted any attention. I have since found that the curves T do not precisely represent the reduced forms of Gauss, but instead a system of forms determined by a different regulative principle. I am disposed to think (notwithstanding the considerations mentioned in the note on Art. 1), that there is some advantage in continuing to regard the equation between the squares of the modules, as the principal modular equation, rather than either the equation (1) or (7). At least, as far as concerns the arithmetical theory to which the Mémoire relates, and which I have since extended to the equations (1), (2), and (7), (the theory of the equations (3), (4), and (6) hardly requiring a separate discussion,) the modular curves (5) present phenomena in some respects simpler than those presented by the curves (7), and in all respects simpler than those presented by the curves (1) ... (4).

Both in the Mémoire and in this paper, I have given especial attention to the case in which N is a square, because the solution of the problem of Jacobi for the transformations of order N depends on a consideration of the spaces into which modular curves of order N^2 divide a plane. A note published in the *Transunti* of the Accademia dei Lincei (Vol. i., p. 42, 7 Jan., 1877), contains what is in fact a solution of Jacobi's problem for the case $N=1$; to this particular case of the general problem, the attention of geometers had been called by M. Hermite in the note appended to the second volume of M. Serret's edition (1862) of the Differential Calculus of Lacroix, pp. 421—425.—H. J. S. S.]

distant is a point). Thus, if we cause q to describe any closed contour, not including one of the three points 0, 1, ∞, the values of p will undergo no interchange; but each root of the equation $F(p, q, 1) = 0$ will return, when the contour is closed, to the same value which it had at the beginning; although the contour may include points (other than 0, 1, ∞) at which two of the values of p become equal. But the case is different if we cause q to describe a closed contour round one of the three exceptional points. At each of these points all the values of p become equal to one another, and to the value of q indicated by the point. As the general result is the same for each of the three points, it will suffice to attend to one of them only; for example, to the point $q = 0$. If, then, we cause q to describe a closed contour round 0, the g values of p, or of the expression $\phi^8\left(\frac{g'\omega + 2k}{g}\right)$, which contain the divisor g in the denominator, change into one another cyclically, and thus the $\Sigma g = A + B$ roots of the equation arrange themselves in 2ν cycles, corresponding to the different divisors of N; or, which is the same thing, the developments obtained by expanding the different values of p in series proceeding by ascending powers of q are singular; being, in fact, of the type

$$(5) \ldots\ldots\ldots p = \lambda q^{\frac{g'}{g}} + \ldots.$$

Similarly, at the points $+1$ and ∞, we have singular developments of the types

$$(6) \ldots\ldots p - 1 = \lambda (q-1)^{\frac{g'}{g}} + \ldots,$$

$$(7) \ldots\ldots \frac{1}{p} = \lambda \left(\frac{1}{q}\right)^{\frac{g'}{g}} + \ldots.$$

As these are all the singular developments that can exist, we infer that, if $W(p)$ represent the number of the spiral points of p, each point being reckoned with its proper multiplicity,

$$W(p) = 3\Sigma(g-1) = 3(A+B) - 6\nu.$$

Substituting, in the equation of Riemann,

$$(8) \ldots\ldots 2H = W(p) - 2E(p) + 2,$$

the value of $E(p)$ (equation viii., Art. 3), and the value just obtained for $W(p)$, we find $H = \frac{1}{2}(A+B) - 3\nu + 1$,

which is the equation (iii.)

Art. 5. *The Cuspidal Index.*—To determine the cuspidal index of C, we first consider the developments (5) which appertain to the point R. Since N is not a square, we cannot have $g = g'$; if $g' > g$, we have the

normal development*

$$(9) \ldots\ldots \frac{\alpha}{\gamma} = \lambda \left(\frac{\beta}{\gamma}\right)^{\frac{g'}{g}} + \ldots;$$

if $g > g'$, we have the normal development

$$(10) \ldots\ldots \frac{\beta}{\gamma} = \mu \left(\frac{\alpha}{\gamma}\right)^{\frac{g}{g'}} + \ldots.$$

The branches corresponding to the developments (9) and (10) have, for their cuspidal indices, $g-1$ and $g'-1$ respectively. Hence each of the lines QR, PR is touched at the point R by a set of branches of which the aggregate cuspidal index is $B-\nu$.

In the same manner, it will be found that at the point S each of the lines QS, PS is touched by a set of branches having the same cuspidal indices as the branches which touch QR or PR at R.

Lastly, from the developments (7), we deduce normal developments of the types

$$(11) \ldots\ldots \frac{\gamma}{\alpha} = \mu \left(\frac{\beta}{\alpha}\right)^{\frac{g'}{g'-g}} + \ldots, \quad g' > g;$$

$$(12) \ldots\ldots \frac{\gamma}{\beta} = \mu \left(\frac{\alpha}{\beta}\right)^{\frac{g'}{g-g'}} + \ldots, \quad g > g';$$

and from these we infer that the line PQ is touched at each of the points P and Q by a set of branches of which the aggregate cuspidal index is $A-B-\nu$.

Since C can have no other cuspidal branches, we find

$$\begin{aligned} K &= 2(B-\nu)+2(B-\nu)+2(A-B-\nu) \\ &= 2(A+B)-6\nu, \end{aligned}$$

which is the formula (iv.).

Art. 6. *The discriminant of* $F(p, q, 1)$.—The values of m, $E(p)$, $E(q)$, $E'(p)$, $E'(q)$ are inferred from the equation (1), by a method, due (as it would seem) to M. Kronecker, of which examples are given in the Report on the Theory of Numbers (*Reports of the British Association* for 1865, p. 349 sqq.) This method is also applicable to the

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* If A, B, C represent straight lines forming a triangle, a development of the type

$$\frac{A}{C} = \lambda_1 \left(\frac{B}{C}\right)^{\alpha_1} + \lambda_2 \left(\frac{B}{C}\right)^{\alpha_2} + \ldots,$$

in which $\alpha_1, \alpha_2 \ldots$ are positive and increasing, and α_1 is greater than unity, is termed a normal development; A is, of course, the tangent to the branch, B is any line passing through the point to which the development refers.

discriminant of the equation (1)*; *i.e.*, to the expression

$$(13) \ldots\ldots \nabla(q) = \Pi[\phi^8(\Omega_1) - \phi^8(\Omega_2)]^2,$$

where the sign of multiplication extends to every pair of values of Ω.

Let $$(14) \ldots\ldots (A+B)^2 = A_2 + B_2,$$

where A_2 comprises all the terms in $(A+B)^2$ which are greater than N, and B_2 all the terms which are less than N; as for the terms which are equal to N (of which the sum is evidently $2\nu N$), we divide them equally between A_2 and B_2; thus, if $g_1, g_2, \ldots g_t$ are all the divisors of N (unity and N included), we have

$$(15) \ldots\ldots \begin{cases} A_2 = \nu N + \sum_{r=1}^{r=t} \sum_{s=1}^{s=t} g_r g_s, & g_r g_s > N; \\ B_2 = \nu N + \sum_{r=1}^{r=t} \sum_{s=1}^{s=t} g_r g_s, & g_r g_s < N; \\ A_2 + B_2 = \sum_{r=1}^{r=t} \sum_{s=1}^{s=t} g_r g_s = (A+B)^2. \end{cases}$$

Applying the method of M. Kronecker, we find that the highest power of q in $\nabla(q)$ is $2A_2 - A - B$, and that the lowest power of q is $2B_2 - A - B$. By a known property of the modular equation,

$$F(1-p, 1-q, 1) = F(p, q, 1);$$

hence $\nabla(q)$ must be divisible by $1-q$ as often as by q; we have, therefore,

$$(16) \ldots\ldots \nabla(q) = (q-q^2)^{2B_2 - A - B} \times \chi(q);$$

where $\chi(q)$ is a rational and integral function of q, not divisible by q or $1-q$, and of the order $2A_2 - 4B_2 + (A+B)$.

By another property of the modular equation we have the identity

$$(pq)^{A+B} \times F\left(\frac{1}{p}, \frac{1}{q}, 1\right) = F(p, q, 1).$$

If therefore we write $p \div p'$ for p, and $q \div q'$ for q, the dialytic discriminant of the bipartite binary quantic

$$(p'q')^{A+B} \times F\left(\frac{p}{p'}, \frac{q}{q'}, 1\right)$$

is symmetric with regard to q and q'; *i.e.*, it is of the form

$$(17) \ldots\ldots [qq'(q'-q)]^{2B_2 - A - B} \times \chi(q, q'),$$

where $\chi(q, q')$ is symmetric and of the order $2A_2 - 4B_2 + A + B$.

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* It has been applied by M. Koenigsberger (*Vorlesungen über die Theorie der Elliptischen Funktionen*, Vol. II., p. 154) to the discriminant of the modular equations between u and v, in the case in which N is not divisible by any square; the result had already been given by M. Hermite in his Mémoire sur la Théorie des Équations Modulaires (*Comptes Rendus*, Vol. xlviii., p. 1079.) The discriminant of the modular equation between $\chi^3(\omega)$ and $\chi^3(\Omega)$ has been similarly treated by M. Krause (*Math. Annalen*, Vol. xii., pp. 1—3).

It will be observed that the order of the discriminant (17) is

$$\begin{aligned} &3(2B_2-A-B)+2A_2-4B_2+A+B \\ &\quad=2(A_2+B_2)-2(A+B) \\ &\quad=2(A+B)(A+B-1), \end{aligned}$$

as it ought to be; and that the equation $\nabla(q)=0$ is to be regarded as having $2B_2-A-B$ infinite roots, and as having lost the same number of dimensions.

We may add that $\chi(q)$ is a perfect square. For, if $q-q_0$ be any factor of $\chi(q)$, all the developments of p, proceeding by powers of $q-q_0$, have integral exponents; the exponent of $q-q_0$ in $\chi(q)$ is the sum of the discriminantal indices of these developments taken in pairs; and this sum is always even.

Art. 7. *The discriminantal indices of* P, Q, R, S.—We next examine the discriminantal indices in the curve C of the points P, Q, R, S. Representing these indices by $D(P)$, $D(Q)$, $D(R)$, $D(S)$, we have

$$(18)\ \ldots\ldots \begin{cases} D(P)=2A^2-A_2-A=D(Q), \\ D(R)=2B_2-2A \quad\ =D(S). \end{cases}$$

To establish these formulæ, it will suffice to consider the points P and R. At P we have ν superlinear branches of the aggregate order $A-B$ touching PQ; we may symbolize the branch of which (11) is the normal development by (g', g), where $gg'=N$, and $g'>\sqrt{N}>g$. The discriminantal index of the branch (g', g) taken by itself is $g'(g'-g-1)$; the joint discriminantal index of the two branches (g', g) and (g_1', g_1) is $2g'(g_1'-g_1)$, if $g'>g_1'$, and consequently $g<g_1$. Hence we have

$$(19)\ \ldots\ldots D(P)=\Sigma g'(g'-g-1)+2\Sigma\Sigma g'(g_1'-g_1),$$

the summations extending to all values of g' and g_1' which satisfy the inequalities $\quad g'>\sqrt{N},\quad g_1'<g',\quad g_1'>\sqrt{N}.$

Attending to these inequalities, we find

$$\Sigma g'^2+2\Sigma\Sigma g'g_1'=A^2,$$

$$\Sigma g'=A,$$

$$\Sigma g'g+2\Sigma\Sigma g'g_1=A_2-A^2;$$

and substituting these values in (19), we obtain the value of $D(P)$, given by the formula (18).

Again, at the point R, we first consider the branches touching PR, the normal developments of which are of the type (10). For the discriminantal index of these branches, taken singly and in pairs, we have the expression

$$(20)\ \ldots\ldots \Sigma g(g'-1)+2\Sigma\Sigma g_1g',$$

the summations extending to all divisors g and g_1 which satisfy the inequalities

$$(21) \ldots\ldots g > \sqrt{N}, \quad g_1 < g, \quad g_1 > \sqrt{N}.$$

The discriminantal index of the branches touching QR is evidently the same as that of the branches touching PR; and as the aggregate order of each of the two sets of branches is B, they intersect one another in B^2 points, and the part of $D(R)$ which arises from their crossing one another at R is $2B^2$. We have, therefore,

$$D(R) = 2\Sigma g(g'-1) + 4\Sigma\Sigma g_1 g' + 2B^2;$$

but, attending to the inequalities (21), we also have

$$\Sigma g = A, \quad \Sigma g' g + 2\Sigma g_1 g' = B_2 - B^2,$$

whence, in accordance with (18),

$$D(R) = 2B_2 - 2A.$$

Art. 8.—*The intersections with C of the polar curves of P and Q.*—Since the branches which touch PR at R are of the aggregate class $A-B$, the line PR, considered as a tangent drawn from P, counts $A-B$ times as a tangent at R. Similarly PS counts $A-B$ times as a tangent at S. Again, since the branches which touch PQ at P and at Q, are of the aggregate order $A-B$, and of the aggregate class B, PQ, considered as a tangent drawn from P, counts $A-B+B=A$ times as a tangent at P, and B times as a tangent at Q. Thus the three lines PR, PS, PQ count as

$$(A-B)+(A-B)+(A+B) = 3A-B$$

tangents from P; *i.e.*, no other tangents can be drawn to C from P. Again, the polar curve of P intersects C at P, Q, R, S, in

$$\begin{aligned} D(P)+A &= 2A^2-A_2, \\ D(Q)+B &= 2A^2-A_2+B-A, \\ D(R)+A-B &= 2B_2-B-A, \\ D(S)+A-B &= 2B_2-B-A, \end{aligned}$$

points respectively; or, in all, in $4A^2+4B_2-2A_2-3A-B$ points. The whole number of intersections of C by any one of its first polars is $2A(2A-1)$; hence the polar of P intersects C in

$$2A(2A-1)-(4A^2+4B_2-2A_2-3A-B) = 2A_2-4B_2+A+B$$

points, other than P, Q, R, S. These $2A_2-4B_2+A+B$ intersections correspond in the discriminant of $F(p, q, 1)$ to the factors of $\chi(q)$, of which the aggregate order is the same.

As the intersections other than those at P, Q, R, S correspond to the

factor $\chi(q)$, so also the intersections at R correspond to the factor q^{2B_2-A-B}, and the intersections at S to the factor $(1-q)^{2B_2-A\ B}$. But it is proper to observe that the remaining intersections at P and Q (of which the aggregate number is $4A^2-2A_2+B-A$) surpass the order of the remaining factor $q'^{\,2B_2-A-B}$ of the complete dialytic discriminant; the difference

$$(4A^2-2A_2+B-A)-(2B_2-A-B)$$
$$=2A(2A-1)-2(A+B)(A+B-1)$$

being, as it ought to be, equal to the difference between the number of intersections of C by any one of its polars, and the order of the dialytic discriminant. We reserve, for a future communication to the Society, a complete discussion of the relations which subsist between the exponents of the factors of the dialytic discriminant of any parametric equation, and the corresponding intersections of the locus curve by the polars of the centres of the generating pencils.

The points, other than P, Q, R, S, in which C is intersected by the polar of P, are all ordinary double (or it may be multiple) points, free from superlinearity, and having tangents which do not pass through P. For C has no superlinear branches beside those at P, Q, R, S, and the only tangents which pass through P are PQ, PR, PS. The same thing is also evident from what has been said in Art. 6 of the exponents of the factors of $\chi(q)$.

Arts. 9—11. *N divisible by a square.*

Art. 9.—*Definition of certain Arithmetical Functions.*—We now pass to the general case in which N is any uneven number whatever. Let

$$N = a_1^{a_1} a_2^{a_2} a_3^{a_3} \ldots,$$

$a_1, a_2 \ldots$ being different uneven primes; let g, g', as before, be two conjugate divisors of N, and let η be the greatest common divisor of g and g'. We resolve g into the product of two factors γ_1 and γ_2, of which γ_1 contains only those prime divisors of g which do not occur in η and g'; and γ_2 contains only those prime divisors of g which do occur in η and g'. Representing by $f(z)$ the number of numbers prime to any given number z and not surpassing it, we write

$$f'(g) = \gamma_1 f(\gamma_2),$$

and we observe that we have the equations

$$\frac{f(\eta)}{\eta} = \frac{f'(g)}{g} = \frac{f'(g')}{g'};$$

each of these quotients being equal to

$$\Pi\left(1-\frac{1}{\epsilon}\right),$$

if ϵ denote any prime divisor of η. We still retain the symbols ν, A, B, A_2, B_2; but with extended significations, which we proceed to explain.

We define ν by the equation

$$(20) \ldots\ldots 2\nu = \Sigma f(\eta),$$

the sign of summation extending to every divisor g of N. We observe that, in general, each term $f(\eta)$ occurs twice in $\Sigma f(\eta)$, because η is the same for each of any two conjugate divisors; but that, if $N = \theta^2$ is a perfect square, the term $f(\theta) = f'(\theta)$ occurs only once in $\Sigma f(\eta)$.

Again, we define A and B by the equations

$$(21) \ldots\ldots A = \Sigma f'(g),\ g \geqq \sqrt{N},$$

$$(22) \ldots\ldots B = \Sigma f'(g),\ g \leqq \sqrt{N},$$

the summations extending to all divisors g of N which satisfy the inequalities (21) and (22) respectively; when $N = \theta^2$ is a perfect square, we divide the term $f'(\theta) = f(\theta)$ equally between A and B. We thus have in every case

$$A + B = \Sigma f'(g),$$

the summation extending to every divisor g of N.

Lastly, we define A_2 and B_2 by the equations

$$(23) \ldots\ldots A_2 = \Sigma\Sigma f'(g_1) f'(g_2),\ g_1 g_2 \geqq N,$$

$$(24) \ldots\ldots B_2 = \Sigma\Sigma f'(g_1) f'(g_2),\ g_1 g_2 \leqq N;$$

in which g_1 and g_2 are any two divisors of N (the same or different) which satisfy the inequalities specified; so that, if g_1 and g_2 are different, the term $f'(g_1) f'(g_2)$ occurs twice in A_2, or in B_2, as the case may be. If $g_1 g_2 = N$, we divide the double term $2f'(g_1) f'(g_2)$, corresponding to these two divisors, equally between A_2 and B_2; if, in particular, $N = \theta^2$ is a perfect square, the single term $[f(\theta)]^2$ is to be divided equally between A_2 and B_2. It is evident that we have, in every case,

$$A_2 + B_2 = (A + B)^2.$$

The sums $2\nu = \Sigma f(\eta)$, and $A + B = \Sigma f'(g)$, may be conveniently expressed in terms of the prime divisors of N. Observing (1) that the terms of the product

$$\Pi [1 + a + a^2 + \ldots a^e]$$

represent, after development, all the divisors of N, and (2) that

$$f'(g) = f'(h_1) \times f'(h_2),$$

if g be a product of two relatively prime factors h_1 and h_2, we find

$$\begin{aligned}\Sigma f'(g) &= \Pi[f'(1)+f'(a)+\ldots f'(a^\alpha)]\\ &= \Pi[1+(a-1)+a(a-1)+\ldots+a^{\alpha-2}(a-1)+a^\alpha]\\ &= \Pi[a^{\alpha-1}+a^\alpha];\end{aligned}$$

whence

$$(25)\ \ldots\ldots\ A+B = N\times\Pi\left(1+\frac{1}{a}\right).$$

Again, if we write $f''(g)$ for $f(\eta)$, and give to h_1, h_2 the same signification as before, $f''(g)$ satisfies the equation

$$f''(g) = f''(h_1)\times f''(h_2);$$

whence we infer that

$$\Sigma f(\eta) = \Pi[f''(1)+f''(a)+\ldots+f''(a^\alpha)].$$

First, let $\alpha = 2\mu+1$; then

$$\begin{aligned}&f''(1)+f''(a)+\ldots f''(a^{2\mu+1})\\ &\quad= 2[1+(a-1)+a(a-1)+\ldots+a^{\mu-1}(a-1)] = 2a^\mu.\end{aligned}$$

Secondly, let $\alpha = 2\mu$; then

$$\begin{aligned}&f''(1)+f''(a)+\ldots+f''(a^{2\mu})\\ &\quad= 2[1+(a-1)+a(a-1)+\ldots+a^{\mu-2}(a-1)]+a^{\mu-1}(a-1)\\ &\quad= a^\mu+a^{\mu-1}.\end{aligned}$$

If therefore $N = \Pi b^{2\beta+1}\times\Pi c^{2\gamma}$, where b, ..., c, ..., are different prime numbers, we have

$$(26)\ \ldots\ldots\ \Sigma f(\eta) = 2\nu = \Pi 2b^\beta\times\Pi c^\gamma\left(1+\frac{1}{c}\right).$$

It will be observed that the definitions which we have now given of the symbols ν, A, B, A_2, B_2, coincide, in the case in which N is not divisible by any square, with the definitions of Arts. 3 and 6.

Art. 10. *Case when N is not a square.*—Excluding, for the present, the case in which N is a perfect square, we have to show that, in all other cases, the formulæ of Arts. 3—8 hold without further modification. For brevity, we shall establish only a few of the assertions included in this general statement, as the method to be pursued with regard to all of them is the same.

(i.) If $\Phi(N)$ is the sum of the divisors of N, and e_1, e_2, ... are the primes of which the squares divide N, the order of the irreducible modular equation of order N is

$$\Phi(N)-\Sigma\Phi\left(\frac{N}{e^2}\right)+\Sigma\Phi\left(\frac{N}{e_1^2e_2^2}\right)-\ldots$$

(See M. Joubert, *Comptes Rendus*, Vol. l., p. 1041; Report on the

Theory of Numbers, *loc. cit.*, p. 332). But this expression has for its value $N\Pi\left(1+\frac{1}{a}\right)$; *i.e.*, the order of the modular equation in p or in q is $A+B$ (equation 25).

(ii.) If we write $q=\phi^8(\omega)$, the $A+B$ corresponding values of p are given by the equation

$$p=\phi^8\left(\frac{g'\omega+2k}{g}\right),$$

in which k is any one of the $\frac{f(\eta)}{\eta}g=f'(g)$ residues of g which are prime to η, the greatest common divisor of g and g'. If, as in Art. 4, we cause q to describe a closed contour round 0, the $f'(g)$ values of p, which answer to any given divisor g, arrange themselves in $f(\eta)$ cycles each containing $\frac{g}{\eta}$ roots; and thus the developments (5), which appertain to the simultaneous values $p=0$, $q=0$, assume, in the general case which we are now considering, the form

$$(27)\ \ldots\ldots\ p=\lambda(q)^{\frac{g'}{\eta}\div\frac{g}{\eta}}+\ldots;$$

the least common denominator of the exponents being $\frac{g}{\eta}$. It will be observed that there are $f(\eta)$ developments, in which g and g' have the same values; the coefficients λ having different values in these $f(\eta)$ developments. Similarly, there are $\Sigma f(\eta)$ developments of each of the types

$$(28)\ \ldots\ldots\ (p-1)=\lambda(q-1)^{\frac{g'}{\eta}\div\frac{g}{\eta}}+\ldots,$$

$$(29)\ \ldots\ldots\ \frac{1}{p}=\lambda\left(\frac{1}{q}\right)^{\frac{g'}{\eta}\div\frac{g}{\eta}}+\ldots.$$

Hence $$W(p)=3\Sigma f(\eta)\left[\frac{g}{\eta}-1\right]$$

$$=3\Sigma f'(g)-3\Sigma f(\eta)=3(A+B)-6\nu;$$

and, consequently, as in Art. 4,

$$H=\tfrac{1}{2}(A+B)-3\nu+1.$$

(iii.) From the developments (27), (28), (29), we can deduce the normal developments of the six sets of branches which touch PR and QR at R, PS and QS at S, PQ at P and Q. Each set comprises ν branches; if h^2 is the greatest square dividing N, h of these are linear in each of the first four sets; all the rest are superlinear. It will suffice here to determine the cuspidal and discriminantal

indices of the branches touching PQ at P. The normal developments of these branches are of the type

$$(30) \ldots\ldots \begin{cases} \frac{\gamma}{\alpha} = \mu \left(\frac{\beta}{\alpha}\right)^{\frac{g'}{\eta} + \frac{g'-g}{\eta}} + \ldots \\ g' > g. \end{cases}$$

Hence their aggregate cuspidal index is

$$\Sigma f(\eta) \left(\frac{g'}{\eta} - \frac{g}{\eta} - 1\right), \quad g < \sqrt{N};$$

or, which is the same thing,

$$g > \sqrt{N}, \quad g < \sqrt{N}, \quad g < \sqrt{N},$$
$$\Sigma f'(g) - \Sigma f'(g) - \Sigma f(\eta) = A - B - \nu.$$

To obtain the discriminantal index, we first consider a single group of $f(\eta)$ branches corresponding to given values of g, g'. The discriminantal index of one of these branches, taken by itself, is

$$\frac{g'}{\eta}\left(\frac{g'}{\eta} - \frac{g}{\eta} - 1\right);$$

the joint discriminantal index of two different branches of the group is

$$2\,\frac{g'}{\eta}\left(\frac{g'}{\eta} - \frac{g}{\eta}\right);$$

so that the aggregate discriminantal index of the group (g', g) is

$$f(\eta) \times \frac{g'}{\eta}\left(\frac{g'}{\eta} - \frac{g}{\eta} - 1\right) + f(\eta)\,[f(\eta) - 1] \times \frac{g'}{\eta}\left(\frac{g'}{\eta} - \frac{g}{\eta}\right)$$
$$= f'(g')\,[f'(g') - f'(g) - 1].$$

We next consider the two groups (g', g) and (g_1', g_1) consisting respectively of $f(\eta)$ and $f(\eta_1)$ branches. If $g' > g_1'$, or, which is the same thing, if $g' g_1 > N$, the joint discriminantal index of the two groups is

$$2f(\eta) f(\eta_1) \times \frac{g'}{\eta}\left(\frac{g_1'}{\eta_1} - \frac{g_1}{\eta_1}\right) = 2f'(g')\,[f'(g_1') - f'(g_1)].$$

Thus the aggregate discriminantal index of the branches touching PQ at P is given by the equation

$$D(P) = \Sigma f'(g')\,[f'(g_2') - f'(g) - 1]$$
$$+ 2\Sigma\Sigma f'(g')\,[f'(g_1') - f'(g_1),$$

the summations extending to all values of g' and g_1' which satisfy the inequalities $\ldots\ldots g' > \sqrt{N}, \ g_1' < g', \ g_1' > \sqrt{N}.$

But, as in Art. 7,

$$\Sigma[f'(g')]^2+2\Sigma\Sigma f'(g')f'(g'_1)=A^2,\quad \Sigma f'(g')=A,$$
$$\Sigma f'(g')f'(g)+2\Sigma\Sigma f'(g')f'(g_1)=A_2-A^2;$$

whence, as before, $D(P)=2A^2-A_2-A$.

Art. 11.—*Case when N is a square.*—The case in which $N=\theta^2$ is a perfect square requires separate consideration, because the modular curve of order θ^2 meets the line PQ in $f(\theta)$ points distinct from one another and from P and Q; and again, at each of the points R and S, it has $f(\theta)$ linear branches, of which the tangents are different from one another, and from the lines RP, RQ, SP, SQ. Thus some of the characteristics of the singularities at $PQRS$ are changed; and with them some of the characteristic indices of the curve.

We write θ' for $f(\theta)=f'(\theta)$. It will be found, on referring to Art. 10, i. and ii., that

$$E(p)=E(q)=A+B,\quad m=2A,$$
$$H=\tfrac{1}{2}(A+B)-3\nu+1,$$

as in the case when N is not a square. Again, the cuspidal index of each of the four sets of branches which touch PR and QR at R, PS and QS at S, is, as before, $B-\nu$; but the cuspidal index of the branches at P and Q is $A-B-\nu+\frac{1}{2}\theta'$ instead of $A-B-\nu$. For this index is

$$\underset{g>\sqrt{N},}{\Sigma f'(g')}-\underset{g<\sqrt{N},}{\Sigma f'(g)}-\underset{g<\sqrt{N},}{\Sigma f(\eta)},$$

(see Art. 10, iii.); and

$$\underset{g>\sqrt{N}}{\Sigma f'(g)}=A-\tfrac{1}{2}\theta',$$
$$\underset{g<\sqrt{N}}{\Sigma f'(g)}=B-\tfrac{1}{2}\theta',$$
$$\underset{g<\sqrt{N}}{\Sigma f(\eta)}=\nu-\tfrac{1}{2}\theta'.$$

To find the discriminantal indices of $PQRS$, we denote by $\overline{A}$, $\overline{B}$, $\overline{A_2}$, $\overline{B_2}$ the numbers obtained by omitting in ABA_2B_2 the terms depending on θ; we thus have

$$\overline{A}=A-\tfrac{1}{2}\theta',\quad \overline{B}=B-\tfrac{1}{2}\theta',$$
$$\overline{A_2}=A_2-2\theta'A+\tfrac{1}{2}\theta'^2,$$
$$\overline{B_2}=B_2-2\theta'B+\tfrac{1}{2}\theta'^2.$$

Using these expressions, we find, as in Art. 10, iii.,

$$D(P)=D(Q)=2\overline{A}^2-\overline{A_2}-\overline{A};$$

or, substituting for $\overline{A}$ and $\overline{A_2}$ their values,

$$D(P)=D(Q)=2A^2-A_2-A+\tfrac{1}{2}\theta'.$$

To determine $D(R)$, we have:—(i.) For the discriminantal index of the set of branches touching either PR or QR, $\overline{B}_2-\overline{A}-\overline{B^2}$; (ii.) for the joint discriminantal index of these two sets of branches, $2\overline{B^2}$; (iii.) for the joint discriminantal index of the θ' linear branches, $\theta'(\theta'-1)$; (iv.) for the joint discriminantal index of the linear branches taken with the branches touching either PR or QR, $2\theta'\overline{B}$.

$$\text{Hence } D(R) = D(S) = 2[\overline{B}_2-\overline{A}-\overline{B^2}]+2\overline{B^2}+\theta'(\theta'-1)+4\theta'\overline{B}$$
$$= 2(B_2-A),$$

as in the case when N is not a square.

There is no change in the expressions for the order of $\nabla(q)$, and for the exponents of the factors q and $1-q$ in $\nabla(q)$ (see Art. 6); and these expressions agree with the values which we have obtained for $D(R) = D(S)$, and for $D(P) = D(Q)$. For PR, touching the curve at R, counts as $A-B$ tangents drawn from P; and hence the order of q in the discriminant ought to be, what in fact it is,

$$2(B_2-A)+(A-B) = 2B_2-A-B.$$

And again, PQ, considered as drawn from P, counts as $A-\frac{1}{2}\theta'$ tangents at P, and as $B-\frac{1}{2}\theta'$ tangents at Q. Thus, the number of intersections of C by the polar of P, which lie on the line PQ, is

$$2(2A^2-A_2-A+\tfrac{1}{2}\theta')+(A-\tfrac{1}{2}\theta')+(B-\tfrac{1}{2}\theta') = 4A^2-2A_2+B-A;$$

and this number is, as it ought to be, the excess of $2A(2A-1)$ above the order of $\nabla(q)$; *i.e.*, the excess of the whole number of intersections above the intersections lying on PQ.

Arts. 12—14. *Formulæ applicable to all values of N.*

Art. 12.—If, in the formulæ relating to the case when N is a square, we omit the terms containing the symbol θ' defined by the equation

$$\theta' = f(\theta) = f'(\theta') = f(\surd N),$$

we obtain the corresponding formulæ for the case when N is not a square. We shall henceforward denote by θ' a number which is equal to zero when N is not a square, and which is equal to $f(\surd N)$ when N is a square; and we shall treat the two cases simultaneously, except when it is necessary to call attention to the difference between them.

Art. 13.—*The discriminantal class of the superlinear branches.*—In the paper on the Higher Singularities of Plane Curves* (Arts. 12 and

* Proceedings of the Society, Vol. vi., p. 153.

13), it has been shown that, if d and t are the order and class of a superlinear branch, D and T its discriminantal order and class, we have the equation $T-D=t^2-d^2$.

And again, that if there be a second superlinear branch of the order d' and class t' touching the first, and if we represent by $\overline{T}$ and $\overline{D}$ the joint discriminantal indices (of order and class) appertaining to the two branches, we have the equation

$$\overline{T}-\overline{D}=2(tt'-dd').$$

Combining these two results, we obtain the theorem—

"If any number of branches touch one another at the same point, the difference between the discriminantal order and class of the singularity is equal to the difference between the squares of its order and of its class."

Employing a notation explained in Art. 14, we apply this theorem to determine the discriminantal class of the branches (PPQ), (PQQ), (PRR), (QRR), (PSS), (QSS). We thus find

$$(31)\ \ldots\ldots\ T(PPQ)-D(PPQ)=T(PQQ)-D(PQQ)$$
$$=(B-\tfrac{1}{2}\theta')^2-(A-B)^2.$$

$$(32)\ \ldots\ldots\ T(PRR)-D(PRR)=T(QRR)-D(QRR)$$
$$=T(PSS)-D(PSS)=T(QSS)-D(QSS)$$
$$=(A-B)^2-(B-\tfrac{1}{2}\theta')^2;$$

so that
$$T(PPQ)=T(PQQ)=B_2-B^2-A-\theta'B+\tfrac{1}{2}\theta'+\tfrac{1}{4}\theta'^2$$
$$=\overline{B}_2-\overline{B}^2-\overline{A}$$
$$=D(PRR)=D(QRR)=D(PSS)=D(QSS),$$

and
$$T(PRR)=T(QRR)=T(PSS)=T(QSS)$$
$$=2A^2-A_2-A+\tfrac{1}{2}\theta'$$
$$=2\overline{A}^2-\overline{A}_2-\overline{A}=D(PQQ)=D(PPQ).$$

Art. 14.—*Summary of the results.*—For convenience of reference, we exhibit the preceding results in a tabular form.

Characteristics and Singularities of the Modular Curve C.

I. *Explanation of the symbols.*—

(1) The order of the transformation is the uneven number N.

(2) g and g' are conjugate divisors of N; h^2 is the greatest square dividing N.

(3) η is the greatest common divisor of g and g'.

(4) $f(\eta)$ is the number of numbers not surpassing η and prime to it.

(5) $f'(g)$ and $f'(g')$ are defined by the equation

$$\frac{f'(g)}{g} = \frac{f(\eta)}{\eta} = \frac{f'(g')}{g'}.$$

(6) $\quad 2\nu = \Sigma f(\eta), \quad A+B = \Sigma f'(g), \quad A_2+B_2 = (A+B)^2.$

In these equations the summations Σ extend to all divisors g of N; A comprehends all the terms $f'(g)$ in which $g > \sqrt{N}$, and also, if $N = \theta^2$, the term $\frac{1}{2}\theta' = \frac{1}{2}f(\theta)$; A_2 comprehends all the terms of $\Sigma f'(g_1) \times \Sigma f'(g_2)$, in which $g_1 g_2 > N$, and one-half of every term in which $g_1 g_2 = N$, g_1 and g_2 denoting any two divisors of N, the same or different. The definitions of B and B_2 follow from those of A and A_2.

(7) m, n, K, I, D, T, H denote respectively the order, the class, the cuspidal index, the inflexional index, the discriminantal order, the discriminantal class, and the deficiency of the curve.

(8) The symbol (XXY) or (YXX) denotes a branch, or an aggregate of branches, touching the line XY at the point X.

(9) The symbols $O(XXY)$, $C(XXY)$, $K(XXY)$, $I(XXY)$, $D(XXY)$, $T(XXY)$ denote the order, class, cuspidal index, inflexional index, discriminantal order, discriminantal class of the branches (XXY). The symbols $O(X)$, $K(X)$, $D(X)$, $C(XY)$, $I(XY)$, $T(XY)$ are to be similarly interpreted with regard to the branches which pass through a given point X or touch a given line XY. Lastly, the symbols $D(XXY, XXZ)$ and $T(XXY, XYY)$ denote respectively twice the number of points common to the branches (XXY), (XXZ), and twice the number of tangents common to the branches (XXY), (XYY).

II. *Characteristics of the Curve.**

$$\begin{aligned} m &= 2A, \quad n = 3A-B-\theta', \\ H &= \tfrac{1}{2}(A+B)-3\nu+1, \\ K &= 2(A+B)-6\nu+\theta', \\ I &= 5A-B-6\nu-2\theta', \\ I-K &= 3A-3B-3\theta', \\ D &= 4A^2-5A+B+\theta', \\ T &= (3A-B-\theta')^2-5A+B+\theta', \\ T-D &= (3A-B-\theta')^2-4A^2. \end{aligned}$$

III. *Characteristics of the Special Singularities.*

(i.) *Characteristics of* (PPQ) *and* (PQQ).

$$\begin{aligned} &O(PPQ) = A-B; \quad C(PPQ) = B-\tfrac{1}{2}\theta', \\ &K(PPQ) = A-B-\nu+\tfrac{1}{2}\theta', \quad I(PPQ) = B-\nu, \\ &D(PPQ) = 2A^2-A_2-A+\tfrac{1}{2}\theta', \\ &T(PPQ) = B_2-B^2-A-\theta'B+\tfrac{1}{2}\theta'+\tfrac{1}{4}\theta'^2. \end{aligned}$$

* Several of the formulæ which follow may be more simply expressed by using the symbols $\overline{A}, \overline{B}, \overline{A_2}, \overline{B_2}$ of Art. 11, and by writing $\overline{\nu} = \nu - \frac{1}{2}\theta'$.

The number of distinct branches is $\nu - \frac{1}{2}\theta'$. They are all superlinear; viz., corresponding to every divisor g of N, which is less than $\sqrt{N}$, there are in (PPQ) $f(\eta)$ superlinear branches, each of the order $\frac{g'-g}{\eta}$, and of the class $\frac{g}{\eta}$; (PQQ) is of the same type as (PPQ).

(ii.) *Characteristics of* (PRR), (QRR), (PSS), (QSS).

All these singularities are of the same type.

$$\begin{aligned}
O(PRR) &= B - \tfrac{1}{2}\theta'; \quad C(PRR) = A - B, \\
K(PRR) &= B - \nu, \\
I(PRR) &= A - B - \nu + \tfrac{1}{2}\theta', \\
D(PRR) &= B_2 - B^2 - A - \theta' B + \tfrac{1}{2}\theta' + \tfrac{1}{4}\theta'^2, \\
T(PRR) &= 2A^2 - A_2 - A + \tfrac{1}{2}\theta'.
\end{aligned}$$

The number of distinct branches in (PRR) is $\nu - \frac{1}{2}\theta'$; of these, $h - \theta'$ are linear (Art. 10, iii.); the characteristics of (PRR) and (PPQ) are reciprocal; viz., corresponding to any divisor g of N, which is less than $\sqrt{N}$, there are in (PRR) $f(\eta)$ branches, each of the order $\frac{g}{\eta}$ and of the class $\frac{g'-g}{\eta}$.

(iii.) *Characteristics of* (PQ).

$$\begin{aligned}
C(PQ) &= 2C(PPQ) = 2B - \theta'; \\
I(PQ) &= 2I(PPQ) = 2B - 2\nu, \\
T(PQ) &= T(PPQ) + T(PQQ) + T(PPQ, PQQ) \\
&= 2T(PPQ) + 2(B - \tfrac{1}{2}\theta')^2 \\
&= 2B_2 - 2A - 4\theta' B + \theta'(\theta' + 1).
\end{aligned}$$

(iv.) *Characteristics of* (R) *and* (S).

These are the same for the two points.

$$\begin{aligned}
(1) \ldots\ldots\ O(R) &= O(PRR) + O(QRR) + O(\theta) \\
&= 2B. \\
(2) \ldots\ldots\ K(R) &= K(PRR) + K(QRR) \\
&= 2B - 2\nu. \\
(3) \ldots\ldots\ D(R) &= D(PRR) + D(QRR) + D(PRR, QRR) \\
&\qquad + D(\theta) + D(\theta, PRR) + D(\theta, QRR) \\
&= 2\left[B_2 - B^2 - A - \theta' B + \tfrac{1}{2}\theta' + \tfrac{1}{4}\theta'^2\right] + 2(B - \tfrac{1}{2}\theta')^2 \\
&\qquad + \theta'(\theta' - 1) + 4\theta'(B - \tfrac{1}{2}\theta') \\
&= 2(B_2 - A).
\end{aligned}$$

The symbol (θ) is used to represent the θ' linear branches which pass through R, having tangents distinct from one another and from PR, QR.

(v.) *Tangents to the Curve from PQRS.*

(1) PQ, considered as drawn from P, counts as $A-\frac{1}{2}\theta'$ tangents at P, and as $B-\frac{1}{2}\theta'$ tangents at Q; PR counts as $A-B$ tangents at R; thus $PQ.\ PR,\ PS$ count as $3A-B-\theta'=n$ tangents drawn from P.

(2) The tangents to the branches (PRR), (QRR), and (θ) count as $2(A-\frac{1}{2}\theta')+2\theta'=2A+\theta'$ tangents drawn from R. Thus, there are $A-B-2\theta'$ other tangents which can be drawn to the curve from R.*

(vi.) *Intersections with the Curve of the sides of the quadrangle PQRS.*

(1) PQ meets the curve in $A-\frac{1}{2}\theta'$ points at P, and in as many at Q; and in θ' non-singular points distinct from either P or Q.

(2) PR meets the curve in $A-B$ points at P; at R it meets the branches (PRR) in $A-\frac{1}{2}\theta'$ points; the branches (QRR) in $B-\frac{1}{2}\theta'$ points; the branches (θ) in θ' points; in all in $2A$ points. The same statements hold, *mutatis mutandis*, for the lines $QR,\ PS,\ QS$.

(3) RS meets the curve $2(B-\frac{1}{2}\theta')+\theta'=2B$ points at R, and in as many at S; and also in $2(A-2B)$ other points.†

IV. *Residual singularities of the Curve.*

Designating by $K_1 I_1 D_1 T_1$ the parts of the indices $KIDT$ which arise from the singularities connected with the points and lines of the quadrangle $PQRS$, we find, from the preceding formulæ,

$$\begin{cases} K_1 = 2(A+B)-6\nu+\theta', \\ I_1 = 4A-2B-6\nu+2\theta', \\ D_1 = 4A^2+4B_2-2A_2-6A+\theta', \\ T_1 = 8A^2+2B_2-4A_2-6A-4\theta'B+3\theta'+\theta'^2; \end{cases}$$

and for the residual singularities we have

$$\begin{cases} K_2 = 0, \\ I_2 = A+B-4\theta', \\ D_2 = 2A_2-4B_2+A+B, \\ T_2 = 4A^2+4B^2+A_2-5B_2+6\theta'(B-A)+A+B-2\theta'. \end{cases}$$

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* If $\chi\left(\frac{1}{M},\ q\right)=0$ represent the equation of the multiplier, which is of the order $A+B$ in $\frac{1}{M}$, and of the order $\frac{1}{2}(A-B)$ in q, the values of q appertaining to the points of contact of these tangents are determined by the equations

$$\chi\left(\sqrt{n},\ \frac{q}{q-1}\right)=0,\quad \chi\left(-\sqrt{n},\ \frac{q}{q-1}\right)=0;$$

when $N=\theta'^2$, the first of these equations has θ' roots equal to zero, and θ' infinite roots; both these sets of roots are to be rejected.

† At each of these points we have $p=q$. The equation $F(p,p,1)=0$ is divisible by $[p(p-1)]^{2B}$; the remaining roots, which are $2A-4B$ in number, give the intersections of the curve by RS at points other than R and S. These roots may be determined by the method (due to M. Kronecker) described in the Report on the Theory of Numbers, Arts. 131—133.

ARTS. 15, 16.—*Case when N is a square. The Linear Branches* (θ).

Art. 15. The developments appertaining to the θ' linear branches U, which intersect PQ at points other than P and Q, are

$$(33) \ldots\ldots -\frac{1}{p}=\frac{\epsilon^{iu}}{q}+\frac{\epsilon^{iu}(1+\epsilon^{iu})}{2q^2}+\frac{\epsilon^{iu}(1+\epsilon^{iu})(21+11\epsilon^{iu})}{64q^2}+\ldots\ldots,$$

where $u=\frac{2l+1}{\theta}\pi$, $2l+1$ being any term of a system of residues prime to 2θ. We hence obtain the normal developments

$$(34) \ldots\ldots \frac{\beta-\frac{1}{2}\gamma+\epsilon^{iu}(\alpha-\frac{1}{2}\gamma)}{\beta}=5\,(1-\epsilon^{2iu})^{\frac{\gamma^2}{\beta^2}}+\ldots\ldots,$$

so that the tangents of the θ' branches are the lines

$$\beta-\tfrac{1}{2}\gamma+\epsilon^{iu}(\alpha-\tfrac{1}{2}\gamma)=0,$$

which meet one another in the point $\alpha=\beta=\frac{1}{2}\gamma$; *i.e.*, in the point O in which RS intersects ab.

The developments appertaining to the θ' linear branches at R are

$$(35) \ldots\ldots p=\epsilon^{iv}q+\tfrac{1}{2}\epsilon^{iv}(1-\epsilon^{v})\,q^2+\ldots\ldots,$$

where $v=\frac{2h}{\theta}\pi$, h being any term of a system of residues prime to θ;

so that the tangents are $\alpha-\epsilon^{v}\beta=0$,

none of the branches being inflected at R.

Similarly the developments appertaining to the θ' linear branches at S are

$$(36) \ldots\ldots p-1=\epsilon^{iv}(q-1)+\tfrac{1}{2}\epsilon^{iv}(1-\epsilon^{iv})(q-1)^2+\ldots\ldots,$$

and the tangents are $\alpha-\gamma=\epsilon^{iv}(\beta-\gamma)$,

there being no inflexion.

The two sets of tangents at R and S meet PQ in the same points in which it is intersected by the linear branches U; for, if

$$2l+1+2h=(2k+1)\,\theta,$$

we have $u+v=(2k+1)\,\pi$, whence $\epsilon^{iv}=-\epsilon^{-iu}$.

Art. 16. The developments (33—36) may be obtained as follows, with the help of formulæ established in the Report on the Theory of Numbers already cited.

If $\omega=1+\frac{i}{\sigma}$, where σ is positive and increases without limit, $q=\phi^8(\omega)=1-\phi^{-8}(i\sigma)$ increases without limit; and the limit of $q\div 16\epsilon^{\pi\sigma}$ is unity. The corresponding values of p are comprised in the formula

$$\phi^8\left(\frac{g'\left(1+\frac{i}{\sigma}\right)+2k}{g}\right),$$

where g and g' are conjugate divisors of N, and g, g', k have no common divisor. But this expression may be exhibited in a form from which the dimensions of $\phi(\Omega)$, as compared with $\phi(\omega)$, may be inferred; viz., we have (Report, *loc. cit.*, p. 350.)

$$\phi^s\left(\frac{g'\left(1+\frac{i}{\sigma}\right)+2k}{g}\right) = \phi^{-s}\left(\frac{d'i\sigma+2l+1}{d}\right),$$

where d' is the greatest common divisor of $g'+2k$ and g, $d=\frac{N}{d'}$, and $2l+1$ is determined by a certain congruence for the modulus d. In order that the development of $\frac{1}{p}$, in a series proceeding by powers of $\frac{1}{q}$, should correspond to a branch intersecting PQ elsewhere than at P or Q, $\frac{1}{p}$ and $\frac{1}{q}$ must be of the same dimensions. But

$$\text{Lim.}\, p \div \epsilon^{\frac{d'}{d}\pi\sigma} = 1, \quad \text{Lim.}\, q \div \epsilon^{\pi\sigma} = 1;$$

hence $d'=d$, or N is necessarily a square, and $d=d'=\theta$. Since $\theta=d'$ is the greatest common divisor of $g'+2k$ and g, let $g=\lambda\theta$, $g'=\frac{\theta}{\lambda}$; then θ divides $\frac{\theta}{\lambda}+2k$; *i.e.*, $\frac{\theta}{\lambda}$ divides g, g', and $2k$, which are relatively prime. Hence $\lambda=\theta$, $g=\theta^2=N$, $g'=1$. Now there are just θ' values of $2k$ for which θ is the greatest common divisor of θ^2 and $1+2k$; viz., if $2\mu+1$ be any number less than 2θ and prime to θ, the θ' values of $2k$ are included in the formula $2k=(2\mu+1)\,\theta-1$; and it will be found that the congruence determining $2l+1$ is $(2\mu+1)(2l+1)\equiv -1$, mod. θ. Hence we have, for the θ' values of $2k$ which we are considering,

$$\phi^s\left(\frac{\frac{i}{\sigma}+1+2k}{N}\right) = \phi^{-s}\left(i\sigma+\frac{2l+1}{\theta}\right) = \phi^{-s}\left(i\sigma+\frac{u}{\pi}\right).$$

Expanding the values of

$$\frac{1}{q} = -\frac{\phi^s(i\sigma)}{1-\phi^s(i\sigma)} \quad \text{and of} \quad \frac{1}{p} = \phi^s\left(i\sigma+\frac{u}{\pi}\right)$$

by means of the formula

$$(37) \ldots\ldots \phi^s(\omega) = 16\epsilon^{i\pi\omega}(1-8\epsilon^{i\pi\omega}+44\epsilon^{2i\pi\omega}-\ldots\ldots),$$

which arises from the expansion of (3); and equating the coefficients of like powers of $\epsilon^{-\pi\sigma}$ in the series

$$\frac{1}{p} = \frac{A}{q}+\frac{B}{q^2}+\frac{C}{q^3}+\ldots\ldots$$

we obtain the developments (33).

Similarly the developments (35), which appertain to the linear branches at R, may be obtained by substituting the expansions of $p = \varphi^8\left(i\sigma + \frac{v}{\pi}\right)$ and $q = \phi^8(i\sigma)$ in the assumed series

$$p = Aq + Bq^2 + \ldots\ldots$$

ARTS. 17—19. *The Six Modular Curves.*

Art. 17. If we represent by $\epsilon(x)$ any one of the six anharmonic functions

$$(38) \quad \ldots\ldots \; x, \quad 1-x, \quad \frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \frac{x-1}{x},$$

the modular equation (1) is unchanged by the simultaneous substitution of $\epsilon(p)$ for p and $\epsilon(q)$ for q. Hence, if $\epsilon_1(x)$, $\epsilon_2(x)$ denote any two, the same or different, of the functions (38), the thirty-six substitutions

$$(39) \quad \ldots\ldots \; F[\epsilon_1(p), \; \epsilon_2(q), \; 1]$$

give only six different equations. As representatives of these, we take the following

$$\text{(i.)} \quad \ldots\ldots \; F(1-p, \; q, \; 1) = 0,$$

$$\text{(ii.)} \quad \ldots\ldots \; F\left(p, \; \frac{1}{q}, \; 1\right) = 0,$$

$$\text{(iii.)} \quad \ldots\ldots \; F\left(1-p, \; \frac{1}{1-q}, \; 1\right) = 0,$$

$$\text{(iv.)} \quad \ldots\ldots \; F(p, \; q, \; 1) = 0,$$

$$\text{(v.)} \quad \ldots\ldots \; F\left(1-p, \; \frac{1}{q}, \; 1\right) = 0,$$

$$\text{(vi.)} \quad \ldots\ldots \; F\left(p, \; \frac{1}{1-q}, \; 1\right) = 0.$$

The equation $F(p, q, 1) = 0$ is symmetric with regard to p and q; and it will be found that the equations (i.), (ii.), (iii.) possess the same property; thus, for example, the equations

$$F\left(p, \; \frac{1}{q}, \; 1\right) = 0, \quad \text{and} \quad F\left(q, \; \frac{1}{p}, \; 1\right) = 0,$$

are the same, because

$$F(x, y, 1) = F(y, x, 1) = (xy)^{A+B} F\left(\frac{1}{x}, \; \frac{1}{y}, \; 1\right).$$

The fifth and sixth equations, on the other hand, are changed, each into the other, by the interchange of p and q.

Art. 18. Denoting by X and Y rectangular Cartesian coordinates,

and writing in the equations (i.) ... (vi.),

$$(40) \quad \ldots\ldots \begin{cases} p = \frac{1}{2} + X - iY \\ q = \frac{1}{2} + X + iY, \end{cases}$$

we obtain the equations of six curves, which, in the Mémoire *Sur les Équations Modulaires*, we have called the first, second, third, fourth, fifth, and sixth modular curves. The equations of the first four of these curves are real, as appears from the symmetry of the equations (i.)—(iv.) with regard to p and q; the equations of the fifth and sixth curves are imaginary and conjugate to one another.

The first and fourth curves are each of them symmetric with regard to both axes; the fourth curve is its own inverse (anallagmatic) with regard to each of the two real circles

$$(X \pm \tfrac{1}{2})^2 + Y^2 = 1;$$

and the first curve with regard to each of the two imaginary circles

$$X^2 + (Y \pm \tfrac{1}{2}i)^2 = -1.$$

The second and third curves are symmetric with regard to the axis of X, and symmetric to one another with regard to the axis of Y; the fifth and sixth (imaginary) curves are symmetric with regard to the axis of Y, and symmetric to one another with regard to the axis of X. The second and third curves are the inverses of the first, with regard to the circles $(X-\frac{1}{2})^2+Y^2=1$, $(X+\frac{1}{2})^2+Y^2=1$, respectively; similarly the two imaginary curves are the inverses of the fourth curve with regard to the two imaginary circles

$$X^2 + (Y \pm \tfrac{1}{2}i)^2 = -1.$$

Lastly, the substitution $X = iY'$, $Y = -iX'$ changes the first curve into the fourth, the second into the fifth, the third into the sixth, and *vice versâ*.

These assertions are the geometrical equivalents of the properties of the modular equation stated in Art. 17; it will suffice to verify one of them. The equation of the first modular curve is

$$F(\tfrac{1}{2}-X+iY,\ \tfrac{1}{2}+X+iY,\ 1) = 0;$$

its inverse with regard to the circle $(X+\frac{1}{2})^2+Y^2=1$ is obtained by writing

$$\tfrac{1}{2}-X+iY = \frac{1}{\frac{1}{2}-X'-iY'},$$

$$\tfrac{1}{2}-X-iY = \frac{1}{\frac{1}{2}-X'+iY'},$$

so that

$$\tfrac{1}{2}+X+iY = 1 - \frac{1}{\frac{1}{2}-X'+iY'}.$$

The equation of the inverse curve is therefore

$$F\left(\frac{1}{\frac{1}{2}-X'-iY'},\ 1-\frac{1}{\frac{1}{2}-X'+iY'},\ 1\right)=0;$$

and this is identical with the equation

$$F\left(\tfrac{1}{2}+X'-iY',\ \frac{1}{\frac{1}{2}+X'+iY'},\ 1\right)=0;$$

i.e., with the equation of the second modular curve, because

$$F\left(\frac{1}{1-y},\ 1-\frac{1}{1-x},\ 1\right)=0$$

is identical with $$F\left(x,\ \frac{1}{y},\ 1\right)=0.$$

Art. 19. The equations of the first and fourth modular curves are included in the general equation

$$F(\alpha, \beta, \gamma)=0;$$

viz., to obtain the first curve, we write

$$\alpha=\tfrac{1}{2}-X+iY,\quad \beta=\tfrac{1}{2}+X+iY,\quad \gamma=1;$$

and, to obtain the fourth curve, we write

$$\alpha=\tfrac{1}{2}+X-iY,\quad \beta=\tfrac{1}{2}+X+iY,\quad \gamma=1.$$

Thus the theory of the singularities of these two curves is implicitly contained in the preceding discussion of the singularities of C.

In both curves the points P, Q are the cyclic points, and (ab, RS) or O is the origin: in the first curve ab and RS are the axes of X and Y; a, b being the points $(\pm\frac{1}{2}, 0)$, and R, S the points $(0, \pm\frac{1}{2}i)$: c is the point at an infinite distance on the axis of Y; in the fourth curve R, S are the points $(\pm\frac{1}{2}, 0)$, and a, b the points $(0, \pm\frac{1}{2}i)$, c being the point at an infinite distance on the axis of X. Both equations (as has been already said) are real; and it follows, from the theory explained in the Mémoire cited, that both of them represent real curves, except when $N\equiv 3$, mod. 4; in which case the fourth curve reduces itself to the pair of conjugate points $(\pm\frac{1}{2}, 0)$.

When N is not a square, both curves are completely and parabolically cyclic, having at each cyclic point ν branches, of the aggregate order $A-B$ and class B, touching the line at an infinite distance.

When N is a square, each of the two curves has θ' real infinite branches. The fourth curve has also θ' real branches passing through each of the points $(\pm\frac{1}{2}, 0)$; (these two points always belong to the curve, though, when N is not a square, only as isolated points:) the

tangents to the θ' branches are parallel to the asymptotes of the curve. Similarly the first modular curve acquires θ' linear, but imaginary, branches at each of the points $(0, \pm\frac{1}{2}i)$; the tangents to these branches being imaginary lines parallel to the real asymptotes.

The equations of the asymptotes of the first and fourth curves are respectively

$$(41) \ldots\ldots \begin{cases} Y\cos\frac{u}{2} - X\sin\frac{u}{2} = 0, \\ Y\sin\frac{u}{2} + X\cos\frac{u}{2} = 0; \end{cases}$$

u denoting $\frac{2l+1}{\theta}\pi$, as in Art. 15. And it may be inferred from the developments (33) and (34), given in that Article, that the rectangular hyperbola

$$\left(Y\cos\frac{u}{2} - X\sin\frac{u}{2}\right)\left(Y\sin\frac{u}{2} + X\cos\frac{u}{2}\right) = 5\sin u$$

osculates at an infinite distance the branches asymptotic to the two lines (41).

Lastly, if $v = \frac{2h}{\theta}\pi$, as in Art. 15, the tangents of the fourth curve at the points $(\pm\frac{1}{2}, 0)$ are

$$Y\cos\frac{v}{2} + (X \pm \tfrac{1}{2})\sin\frac{v}{2} = 0;$$

the tangents of the first curve at the imaginary points $(0, \pm\frac{1}{2}i)$ are

$$(Y \pm \tfrac{1}{2}i)\sin\frac{v}{2} - X\cos\frac{v}{2} = 0:$$

and these tangents are parallel to the asymptotes of the curves to which they respectively appertain; because, if

$$2l+1+2h = (2k+1)\theta,$$

$$\tan\frac{u}{2} = \cot\frac{v}{2}.$$

The points $(\pm\frac{1}{2}, 0)$ and $(0, \pm\frac{1}{2}i)$ are foci, and indeed the only foci, of both curves: of these, the points $(0, \pm\frac{1}{2}i)$ lie on the first curve, and the two real points $(\pm\frac{1}{2}, 0)$ are its two foci (properly so called); the axis of Y being the only corresponding cyclic axis, or directrix. The points $(\pm\frac{1}{2}, 0)$ belong to the fourth curve (only as isolated points, when N is not a square), and this curve has, properly speaking, only the pair of imaginary foci $(0, \pm\frac{1}{2}i)$.

Art. 19. The second and third modular curves may be regarded as

derived from the equation (1), by the substitution

$$(42) \ldots\ldots \begin{cases} p = \dfrac{\alpha}{\gamma}, \quad q = \dfrac{\gamma}{\beta}, \\ \alpha = \frac{1}{2} \pm (X - iY), \\ \beta = \frac{1}{2} \pm (X + iY), \\ \gamma = 1. \end{cases}$$

X and Y being rectangular Cartesian coordinates, and the upper signs relating to the second curve, the lower to the third.

Thus the theory of each of these curves is comprehended in that of the curve C', of which the trilinear equation is

$$(43) \ldots\ldots \begin{cases} (\beta\gamma)^{B-A} \times F(\alpha\beta, \gamma^2, \beta\gamma) = 0, \\ \text{or} \\ (\alpha\gamma)^{B-A} \times F(\alpha\beta, \gamma^2, \alpha\gamma) = 0. \end{cases}$$

The singularities of C' may be examined by the method already employed in the case of C. Attending, for brevity, only to the case in which N is not divisible by any square, we write $p = \dfrac{\alpha}{\gamma}$, $q = \dfrac{\gamma}{\beta}$, in the parametric developments of Art. 4, and we deduce, as follows, the normal developments of the singular branches of C'.

(i.) From (6) we obtain

$$\frac{\alpha - \gamma}{\gamma} = \lambda \left(\frac{\gamma - \beta}{\beta}\right)^{\frac{g'}{g}} + \ldots\ldots;$$

or, multiplying by $\dfrac{\gamma}{\beta} = 1 + \dfrac{\gamma - \beta}{\beta}$,

$$(44) \ldots\ldots \frac{\alpha - \gamma}{\beta} = \lambda \left(\frac{\gamma - \beta}{\beta}\right)^{\frac{g'}{g}} + \ldots\ldots,$$

which is itself a normal development, if $g' > g$, and gives rise, by reversion, to such a development, if $g' < g$. Hence C' has a singularity at S, having the same characteristics as the corresponding singularity of C.

(ii.) From (5) we infer

$$\frac{\alpha}{\gamma} = \lambda \left(\frac{\gamma}{\beta}\right)^{\frac{g'}{g}} + \ldots\ldots.$$

Here, when p and q are small, α must be small compared with γ, and γ compared with β; *i.e.*, the coordinates of the point ($p=0$, $q=0$) are $\alpha = 0$, $\gamma = 0$; and the normal development is

$$(45) \ldots\ldots \frac{\alpha}{\beta} = \lambda \left(\frac{\gamma}{\beta}\right)^{\frac{g+g'}{g}} + \ldots\ldots.$$

(iii.) Similarly from the development (7) we deduce

$$\frac{\gamma}{a} = \lambda \left(\frac{\beta}{\gamma}\right)^{\frac{g'}{g}} + \ldots\ldots.$$

Thus the coordinates of the point ($p=\infty$, $q=\infty$) are $\beta=0$, $\gamma=0$; and we find, after reversion and multiplication by $\frac{\gamma}{a}$, the normal development

$$(46) \ldots\ldots \frac{\beta}{a} = \mu \left(\frac{\gamma}{a}\right)^{\frac{g'+g}{g'}} + \ldots\ldots,$$

which is of the same type as (45).

Thus the curve has at P and Q singularities of one and the same self-reciprocal type, not resembling the singularities which C has at the same points. The point R does not lie on C'.

Art. 20.

Characteristics and Singularities of the Modular Curve C'.

I. *Characteristics of the Curve.*

$$\begin{aligned} m &= 2A+2B, \quad n = 3A+B, \\ H &= \tfrac{1}{2}(A+B)-3\nu+1, \\ K &= 2A+4B-6\nu, \\ I &= 5A+B-6\nu, \\ I-K &= 3(A-B), \\ D &= 4(A+B)^2-5A-3B, \\ T &= (3A+B)^2-5A-3B, \\ T-D &= (3A+B)^2-4(A+B)^2 \\ &= (A-B)(5A+3B). \end{aligned}$$

It will be noticed (1) that these formulæ do not contain θ', although the case when N is a square is included in them; (2) that, when N is not a square, $I-K$ has the same value for C' as for C.

II. *Characteristics of the Special Singularities.*

(i.) *Characteristics of* (PPR) *and* (QQR).

$$\begin{aligned} O(PPR) &= A+B = C(PPR), \\ K(PPR) &= A+B-2\nu = I(PPR), \\ D(PPR) &= A_2+3B_2-2A-2B = T(PPR). \end{aligned}$$

The number of distinct branches at each of the points P and Q is 2ν; viz., corresponding to every divisor g of N, there are in (PPR), $f(\eta)$ branches of the order $\frac{g}{\eta}$ and of the class $\frac{g'}{\eta}$; of the 2ν branches, h are

linear, and, in particular, when N is a square, θ' of these are also non-inflexional.

(ii.) *Characteristics of* (PSS), (QSS), *and* (S).

These are the same for C' as for C (see Art. 14, III., ii. and iv.). When N is a square the equations of the tangents to the linear branches are

$$\alpha - \gamma = \epsilon^{i\psi}(\gamma - \beta)$$

(see Art. 15, equation 36).

(iii.) *Tangents to the Curve from* $PQRS$.

(1) PR, considered as a tangent drawn from P, counts as $2A+2B$ tangents at P; and PS counts as $A-B$ tangents; hence PR and PS are the only tangents from P.

(2) RP, considered as a tangent drawn from R, counts as $A+B$ tangents at P; and so does RQ at Q; thus there are $A-B$ other tangents which can be drawn to C' from R.

(3) Besides the tangents at S, there are $A-B-2\theta'$ other tangents which can be drawn to the curve from S (see Art. 14, III., v., 2).

(iv.) *Intersections with the Curve of the sides of the quadrangle* $PQRS$.

(1) PQ meets the curve in $A+B$ points at P, and in $A+B$ points at Q. Thus it never meets the curve again, and touches it nowhere.

(2) PR and QR each meet the curve in $2A+2B$ points, touching it at P and Q respectively, and meeting it nowhere else.

(3) PS and QS meet the curve in $A+B$ points at S, and in $A+B$ points at P and Q respectively; thus they never meet the curve again.

(4) RS meets the curve in $2B$ points at S, and in $2A$ other points.*

III. *Residual Singularities of the Curve.*

Employing the notation of Art. 14, IV., we have

$$\begin{cases} K_1 = 2A+4B-6\nu, \\ I_1 = 4A-6\nu+\theta', \\ D_1 = 2A_2+8B_2-6A-4B, \\ T_1 = 4A^2+6B_2-6A-4B+\theta'. \end{cases}$$

$$\begin{cases} K_2 = 0, \\ I_2 = A+B-\theta', \\ D_2 = 2A_2-4B_2+A+B, \\ T_2 = 2A^2-2B^2+3A_2-3B_2+A+B-\theta'. \end{cases}$$

The indices K_2, D_2 have the same values for C and C', because these

* The points of contact of the $A-B$ tangents, iii. 2, and of the $A-B-2\theta'$ tangents, iii. 3, and the $2A$ points of intersection, iv. 4, can be determined by methods similar to those indicated in the Notes on Art. 14.

indices refer to singularities which do not lie, in either figure, upon the fundamental triangle of the quadric transformation by which the curves are changed into one another. The equality of the indices I_2, when N is not a square, implies the theorem:

"Each of the first three modular curves has as many non-singular inflexional tangents as it has osculating circles, which pass through the point $(\frac{1}{2}, 0)$; or again, through the point $(-\frac{1}{2}, 0)$."

Of the formulæ contained in the preceding enumeration, we shall demonstrate only one; viz., the expression for $D(PPR)$ or $D(P)$.

We have, as in Art. 10, iii.,

$$D(P) = \Sigma f(\eta)\frac{g+g'}{\eta}\left(\frac{g}{\eta}-1\right) + \Sigma f(\eta)[f(\eta)-1]\frac{g(g+g')}{\eta^2} + 2\Sigma\Sigma_1 f(\eta) f(\eta_1)\frac{g_1(g+g')}{\eta\eta_1};$$

where g is any divisor of N, and g' is the conjugate divisor of g; g_1 is any divisor less than g, so that $g_1' > g'$; the summations Σ_1 and Σ extend to every value of g_1 and g respectively. Hence we find

$$(47) \ldots\ldots D(P) = \Sigma f'(g)[f'(g)+f'(g')] - \Sigma[f'(g)+f'(g')] + 2\Sigma\Sigma_1 f'(g_1)[f'(g)+f'(g')].$$

But we have, evidently,

$$\Sigma f'(g) f'(g) + 2\Sigma\Sigma_1 f'(g_1) f'(g) = (A+B)^2 = A_2+B_2;$$
$$\Sigma[f'(g)+f'(g')] = 2A+2B;$$

and, observing that $g'g_1 < N$, we also find

$$\tfrac{1}{2}\Sigma f'(g) f'(g') + \Sigma\Sigma_1 f'(g_1) f'(g') = B_2.$$

Introducing these values into the equation (47), we obtain

$$D(P) = A_2 + 3B_2 - 2A - 2B,$$

in accordance with the formula II. i. *suprà*.

The following presents were made to the Library in the Vacation:—

"Educational Times," August—November, 1878.

"Reprint of Mathematics from the Educational Times," Vol. xxix.

"Proceedings of the Musical Association for the Investigation and Discussion of Subjects connected with the Art and Science of Music," 4th Session, 1877–8.

"Monatsbericht," Juni, Juli, August, 1878.

"American Journal of Mathematics, Pure and Applied," Vol. i., No. 3; Baltimore, 1878.

"Bulletin des Sciences Mathématiques et Astronomiques," 2me Série, tome 1er; "Table des Matières et Noms d'Auteurs." Also Nos. for Fev., Mars, Avril, Juin, 1878.

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"On the Product $1^1 . 2^2 . 3^3 \ldots n^n$." (Ditto, No. 75, 1877.)

"Series and Products for π and Powers of π." (Ditto, No. 77, 1877.)

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"On Expressions for the Theta Functions as Definite Integrals" ("Proceedings of Cambridge Philosophical Society," Vol. iii., Pt. iii., May 1877).

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"On certain Determinants," and "On a Series Summation, leading to an expression for the Theta Function as a Definite Integral" ("Report of British Association," 1876).

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The above 21 pamphlets from the author, Mr. J. W. L. Glaisher, F.R.S.

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"The analytical Theory of Heat," by J. Fourier; translated, with Notes, by A. Freeman, M.A., Cambridge, 1878: from the Translator.

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"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," redigirt von Dr. Rudolf Wolf; 21er Jahrgang, 1es, 2es, 3es, 4es Heft; 22er Jahrgang, 1es, 2es, 3es, 4es Heft.

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APPENDIX.

The result of Professor Cayley's researches on the Double Theta Functions (p. 29) is given in his Memoir "On the Double Theta Functions" ("Crelle," 85 Band, 3es Heft, pp. 214—245, in continuation of 83 Band, pp. 210—233).

Herr Weichold's solution of the Irreducible Case, of which an abstract was given in Vol. viii. (pp. 312—316), is printed *in extenso* in No. 1 of the "American Journal of Mathematics, Pure and Applied" (pp. 32—49).

We may refer to the same Journal, No. 3 (pp. 261—276), No. 4 (pp. 384, 385), for a very full and nearly complete Bibliography of Hyper-space and non-Euclidean Geometry, by G. B. Halsted (*cf.* Note to Vol. viii., London Mathematical Society's "Proceedings," p. 310).

"On the Theory of Groups" (pp. 126—133), see remarks by Prof. Cayley in the American Journal above cited, No. 1 (pp. 50—52), No. 2 (pp. 174—176); see also "Mathematische Annalen," Band xiii., 4es Heft (pp. 561—565).

The same Journal contains a paper by M. Edouard Lucas, "Théorie des Fonctions Numériques simplement Périodiques" (pp. 184—240, 289—321), bearing upon a subject treated of in the "Proceedings" by Prof. H. J. S. Smith and Mr. Samuel Roberts.

In No. 4 of the same Journal (pp. 350—358), in a paper entitled "Applications of Grassmann's Extensive Algebra," Prof. Clifford gives,

in a brief form, the purport of his once contemplated paper, "On the Classification of Geometric Algebras" (Vol. vii., p. 135).

Prof. Clifford's "Classification of Loci" (Vol. viii., p. 184) was presented as a paper and received by the Royal Society, April 8, 1878. An abstract of its contents is given in Vol. xxvii., No. 187, of the "Proceedings of Royal Society," pp. 370, 371.

Mr. J. C. Malet's paper, "Proof that every Algebraic Equation has a Root" (Vol. viii., p. 289), is printed in the "Transactions of the Royal Irish Academy" (Vol. xxvi., July 1878, No. xiv.)

In connection with Dr. Hirst's "Note on the Correlation of two Planes" (Vol. viii, pp. 262—273), see "Annali di Matematica" (Dec. 1877, pp. 287—300).

Prof. H. J. S. Smith's Presidential Address (Vol. viii., pp. 6—29) is translated by Dr. H. G. Zeuthen, and appears in the "Tidsskrift for Mathematik" (May, 1877).

There is a passage in Chasles' "Aperçu Historique" (p. 125),—"Si à une épicycloïde, engendrée par un point d'une circonférence de cercle qui roule sur un autre cercle fixe, on circonscrit des angles tous égaux entre eux, leurs sommets seront situés sur une épicycloïde allongée ou raccourcée,"—wherein Prof. Wolstenholme's like discovery (Vol. iv., p. 330) is anticipated.

A paper by Sir W. Thomson, "On a Machine for the Solution of Simultaneous Linear Equations," ("Proceedings of Royal Society," Vol. xxviii., No. 191, pp. 111—113), is, if we mistake not, connected with his communication, "The Integration of the Equations for the Motion of a System acted on by Forces expressed by Linear Functions of the Displacements and Velocities" (Vol. vi., p. 114). An account of the recent communication will be found in "Nature" (Vol. xix., pp. 161, 162).

An account of Robert Flower, the logarithmist (p. 75), is given in a letter from Mr. A. J. Ellis to the Editor of the "Academy" ("Academy," April 20, 1878, pp. 347, 348).

In connection with Dr. Klein's paper (p. 123), see his fuller paper in the "Mathematische Annalen." xiv. Band (Heft i., p. 111, to Heft ii., p. 172), entitled "Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades."

On p. 148 the reference, in the text, to "Nature" is incorrectly printed: it should be Vol. xviii., and the form should be $2^{2^m}+1$. Verifications of both M. Pervouchine's results have been presented to the Society by Mr. J. Bridge, M.A. (*cf.* "Nature," Vol. xix., pp. 17, and 73, 74); and a verification of the earlier result has also been made and presented to the Society by the author, Mr. T. Muir, M.A. (*cf.* "Nature," Vol. xviii., p. 652).

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[*The mark * indicates that the Paper has either been printed in extenso, or an abstract of it given in the Proceedings.*]

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ERRATA.

Page 28, line 17, *for* "cross-centre" *read* "mass-centre."

„ 148, „ 22, *read* Vol. xviii.

„ 148, „ 23, *read* 2^{2m}.

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